A CONVERGENCE THEOREM FOR CERTAIN LAGRANGE INTERPOLATION POLYNOMIALS

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In the Lagrange interpolation polynomial $L_n[f;\theta]$ where

$$(1)\quad \begin{aligned} L_n[f;\theta] &\equiv \sum_{k=1}^{n} f(x_k) l_k[\theta], \\ l_k[\theta] &\equiv l_k^{(n)}[\theta] \equiv l_k(x) \equiv \frac{\phi_n(x)}{\phi_n'(x_k)(x-x_k)}, \\ \phi_n(x) &\equiv \prod_{k=1}^{n}(x-x_k), \end{aligned}$$

$$x = \cos\theta;\ -1 < x_k < 1;\ k = 1, 2, \cdots, n;\ n = 1, 2, \cdots,$$

and $f(x)$ is a continuous function defined in $(-1, 1)$, we suppose that

$$(2)\quad x_k \equiv x_k^{(n)} = \cos\theta_k = \cos k\pi/(n+1).$$

Then [1],[1] we have

$$(3)\quad \begin{aligned} \phi_n(x) &= \frac{\sin(n+1)\theta}{2^n \sin\theta}, \qquad x = \cos\theta, \\ l_k[\theta] &= \frac{(-1)^{k+1}\sin^2\theta_k \sin(n+1)\theta}{(n+1)\sin\theta(\cos\theta - \cos\theta_k)}. \end{aligned}$$

We introduce the following notations:

$$(4)\quad \begin{aligned} t_n \equiv t \equiv \theta_1/2 \equiv \pi/2(n+1), \qquad M &= \max_{-1\leqq x\leqq 1} |f(x)|, \\ S_k[\theta] &\equiv \{l_k[\theta - t] + l_k[\theta + t]\}/2. \end{aligned}$$

We shall prove the following theorem which was suggested by a similar theorem of Grünwald [2].

THEOREM. *Let $f(x)$ be a continuous function in the interval $-1\leqq x\leqq 1$. Then*

$$(5)\quad \lim_{n\to\infty} (1/2)\{L_n[f;\theta - t_n] + L_n[f;\theta + t_n]\} = f(\cos\theta), \quad 0 < \theta < \pi,$$

and the convergence is uniform in the interval $0<\alpha\leqq\theta\leqq\pi-\alpha$ (α arbi-

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[1] The numbers in brackets refer to the bibliography.