THE ZEROS OF CERTAIN COMPOSITE POLYNOMIALS

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1. **Introduction.** If $A_0(z)$ is a given mth degree polynomial and

$$(1.1)\quad A_k(z) = (\beta_k - z)A'_{k-1}(z) + (\gamma_k - k)A_{k-1}(z), \qquad \gamma_k \neq m + k, \quad k = 1, 2, \cdots, n,$$

we may obtain various theorems on the relative location of the zeros of $A_0(z)$ and $A_n(z)$ by the familiar method of first finding such relations for two successive $A_k(z)$ and then iterating the relations n times.

This method has already been employed in the study of the zeros of sequence (1.1) for the following three cases: (1) for all k, $\beta_k=0$ and γ_k is real;[1] (2) for all k, $\gamma_k=m+1$—a limiting case leading to Grace's theorem,[2] and (3) the limiting case that for all k, as $h\to 0$, $h\beta_k\to\beta_k'$ and $h(\gamma_k-k)\to 1$, in which case lim $h^kA_k(z)$ is a linear combination of $A_0(z)$ and its first k derivatives.[3]

In the present article we propose to apply the method to the case that *the parameters β_k and γ_k are complex numbers represented by points within certain given regions of the plane.*

To calculate the nth iterate $A_n(z)$ in our case, let us define

$$(1.2)\quad A(z) \equiv A_0(z) \equiv a_0 + a_1 z + \cdots + a_m z^m;$$

$$(1.3)\quad \begin{aligned} B(z) &\equiv (\beta_1 - z)(\beta_2 - z)\cdots(\beta_n - z) \\ &\equiv b_0 + b_1 z + \cdots + b_n z^n, \end{aligned}$$

$$(1.4)\quad C(z) \equiv (\gamma_1 - 1 - z)(\gamma_2 - 2 - z)\cdots(\gamma_n - n - z);$$

$$S(z, k, p) \equiv B(z)\sum \frac{\gamma_{j_1}^{(k+p)} - 1}{\beta_{j_1} - z}\cdot\frac{\gamma_{j_2}^{(k+p)} - 2}{\beta_{j_2} - z}\cdots\frac{\gamma_{j_{n-p}}^{(k+p)} - (n-p)}{\beta_{j_{n-p}} - z},$$

where $[\gamma_j^{(r)}\equiv\gamma_j - r]$ thus $\gamma_j^{(r)}-j$ is a zero of $C(z+r)$, $p<n$, and the sum is formed for all j_i such that $1\leqq j_1<j_2<\cdots<j_{n-p}\leqq n$;

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[1] See Laguerre, *Oeuvres*, Paris, 1898, vol. 1 pp. 200–202, and G. Polya, *Ueber einem Satz von Laguerre*, Jber. Deutschen Math. Verein. vol. 38 (1929) pp. 161–168.

[2] See Laguerre, *Oeuvres*, vol 1 p. 49, and G. Szegö, *Bemerkungen zu einem Satz von S. H. Grace*, Math. Zeit. vol. 13 (1922) pp. 28–55, p. 33.

[3] See M. Fujiwara, *Eine Bemerkungen uber die elementare Theorie der algebraischen Gleichungen*, Tôhoku Math. J. vol. 9 (1916) pp. 102–108; T. Takagi, *Note on the algebraic equations*, Proceedings of the Physico-Mathematical Society of Japan vol. 3 (1921) pp. 175–179; J. L. Walsh, *On the location of the roots of polynomials*, Bull. Amer. Math. Soc. vol. 30 (1924) p. 52, and M. Marden, *On the zeros of the derivative of a rational function*, Bull. Amer. Math. Soc. vol. 42 (1936) p. 406.