BOOK REVIEWS

Sur les ensembles de distances des ensembles de points d'un espace Euclidien. (Mémoires de l'Université de Neuchatel, vol. 13.) By Sophie Piccard. L'Université de Neuchatel, 1939. 212 pp.

Let A denote a set of points on the real number-line R. The set of all number $|a-a'|$, for any a and a' in A, is the set of distances of the set A, and is denoted by $D(A)$.

A connection between A and $D(A)$ arises naturally in the following way. Let r be a real number, and let $x \rightarrow y = x + r$ be a translation of R carrying x into y. Let this translation carry A into $A(r)$. A number z will be in both A and $A(r)$ if and only if z is in A and also at a directed distance r from some point of A. Thus the intersection $A \cdot A(r)$ contains exactly one point for each distinct unordered pair (a, a') of elements of A such that $|a-a'| = |r|$. It is not surprising, then, to find many theorems based on the cardinal number of those r for which the cardinal number of $A \cdot A(r)$ is restricted in some way.

This particular connection is of largely auxiliary interest, however, for the main purpose of the present memoir is to examine for their own sake the relations between properties of A and $D(A)$. The results are so numerous and so detailed that we are obliged merely to cite a few typical items from each of the four chapters. Definitions and notation are as follows: A is congruent to B $(A \cong B)$ when and only when a real number k can be found such that one of the transformations $x \rightarrow y = \pm x + k$ of R carries A into B. The complement of A in R is denoted by CA. The complement of $D(A)$ in the set $0 \leqq x < \infty$ is denoted by $D'(A)$ (reviewer's notation); since 0 is in $D(A)$ naturally, any member of $D'(A)$ is positive. If A and B are two sets, $D(A, B)$ is the set of all numbers $|a-b|$ for any a in A and b in B.

The most important material in the first chapter is a summary of general theorems, due mainly to Sierpinski, whose acknowledged influence appears clearly throughout the book. We learn, for instance, that, if A is open, an F_δ, a denumerable G_δ, Borel-measurable and with $A \cdot A(r)$ at most denumerable for each r, analytic, or of the second Baire category, then $D(A)$ is, respectively, a G_δ, an F_δ, a $G_{\delta\sigma}$, Borel-measurable, analytic, or of the second Baire category. A short proof is given of Steinhaus' important theorem that, if A has positive measure, then $D(A)$ contains an interval $0 \leqq x < k$ for some k. Finally, a partial answer is given to the perfectly natural question: Evidently $A \cong B$ implies $D(A) = D(B)$; when does $D(A) = D(B)$ imply $A \cong B$? If A and B are finite, it is necessary and sufficient to assume