THE CAUCHY THEOREM FOR FUNCTIONS ON CLOSED SETS

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The object of this paper is to extend the theorem of Cauchy to functions of a complex variable defined on any bounded closed set, E, by determining conditions on $f(z)$ in order that for certain coverings of E, C_n, and an extension of $f(z)$, $f^*(z)$, $\lim_{n\to\infty} \int_{c_n} f^*(z)dz=0$. It was suggested partly by the notion of a general monogenic function due to Trjitzinsky[1] and partly by the measure theory methods of Menchoff[2] and others, which succeed so well in lightening the restrictions on the real and imaginary parts of a complex function in order that $f(z)$ be regular.

Throughout this paper we shall consider only rectangles with sides parallel to the real and imaginary axes. A C-covering of a plane set F, denoted by C, will be a set of closed rectangles, possibly abutting, but nonoverlapping, which contain F. c will denote the boundary of C. The covering C_n is to be composed of rectangles R_{mn} so that $C_n=\sum_m R_{mn}$ $(m, n=1, 2, \cdots)$.

1. **The extension,** $f^*(z)$. If $u(P)$ is a positive continuous function defined on the closed and bounded set F in the plane, we shall let[3] $u^*(P)=\max_{Q\in F} u(Q)\{2-d(P, Q)/d(P, F)\}$ for P not in F, and $u^*(P)=u(P)$ for P in F, where $d(P, Q)$ denotes the distance from P to Q and $d(P, F)$ the distance from the set F to P. In general, if $u(P)$ is continuous, since $u(P)=(u(P)+|u(P)|)/2-(|u(P)|-u(P))/2$, that is, since $u(P)$ is the difference of two continuous positive functions, $u^*(P)$ will denote the extension of $u(P)$ obtained by extending as before these parts. If $f(z)$ $(=u(x, y)+iv(x, y))$ is defined on a bounded closed set and continuous, $f^*(z)$ will denote $u^*(x,y)+iv^*(x,y)$.

LEMMA 1. *If $u(P)$ is defined on a bounded closed set F and $|u(Q)-u(P)|<M(P)d(P, Q)$ where $M(P)$ is a finite function of P defined on F, then $|u^*(P)-u^*(Q)|<20\ M(P)\ d(P, Q)$, for P in F and Q arbitrary.*

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[1] W. J. Trjitzinsky, *Théorie des Fonctions d'une Variable Complexe Définies sur des Ensembles Généraux*, Annales Scientifique de L'École Normale Supérieure, Paris, 1938, p. 120.

[2] D. Menchoff, *Les Conditions de Monogénéité*, Actualités Scientifiques et Industrielles, no. 329, Paris, 1936.

[3] S. Bochner, *Fourier Lectures*, 1936–1937, Princeton, p. 62.