DUAL GEODESICS ON A SURFACE

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Introduction. Union curves and dual union curves have been defined and studied in projective space by Sperry.[1] It is well known that the union curves of the congruence of normals to a metric surface are the geodesics on the surface. The principal aim of this note is to obtain the differential equation of the *dual* geodesics on a metric analytic surface in ordinary space.

The notation of Eisenhart[2] will be employed for the most part. However, $\Gamma^\alpha_{\beta\gamma}$ will be used here as the Christoffel symbol of the second kind. Greek indices will take the range 1, 2, and Latin indices the range 1, 2, 3.

1. **Ray-point corresponding to a point of a curve on the surface.** The tangent planes to the surface S $(x^i=x^i(u^1, u^2))$ at the point $P(x^i)$ and at two "successive" points of the curve C $(u^\alpha=u^\alpha(s))$ on S are given by

$$(1)\qquad \begin{aligned} &(\xi^i - x^i)X^i = 0,\\ &(\xi^i - x^i)\frac{\partial X^i}{\partial u^\alpha}u'^\alpha = 0,\\ &(\xi^i - x^i)\left(\frac{\partial^2 X^i}{\partial u^\alpha \partial u^\beta}u'^\alpha u'^\beta + \frac{\partial X^i}{\partial u^\alpha}u''^\alpha\right) = \frac{\partial X^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta}u'^\alpha u'^\beta, \end{aligned}$$

where the primes indicate differentiation with respect to s.

The ray-point[3] R of the curve C corresponding to the point P is the point of intersection of the three planes (1). The coordinates of R are given by

$$(2)\qquad S(\xi^i - x^i) = \delta^{jk}_{\sigma\nu} X^\sigma \frac{\partial X^l}{\partial u^\alpha}\frac{\partial x^l}{\partial u^\beta}\frac{\partial X^\nu}{\partial u^\gamma}u'^\alpha u'^\beta u'^\gamma,$$

where i, j, k take the values 1, 2, 3 cyclically, $\delta^{jk}_{\sigma\nu}$ is a Kronecker delta, and S is defined by

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[1] Sperry, *Properties of a certain projectively defined two-parameter family of curves on a general surface*, American Journal of Mathematics, vol. 40 (1928), p. 213.

[2] Eisenhart, *Differential Geometry*, Princeton University Press, 1940.

[3] Lane, *Projective Differential Geometry of Curves and Surfaces*, The University of Chicago Press, 1932.