SOME NOTES ON AN EXPANSION THEOREM OF PALEY AND WIENER

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Paley and Wiener[1] have formulated a criterion for a set of functions $\{g_n\}$ to be "near" a given orthonormal set $\{f_n\}$. The interest of this criterion is that it guarantees the set $\{g_n\}$ to have expansion properties similar to an orthonormal set.[2] In particular, they show that the set $\{g_n\}$ approximately satisfies Parseval's formula. In the first part of this paper we show that, conversely, if a set $\{g_n\}$ approximately satisfies Parseval's formula then there exists at least one orthonormal set which it is "near."

In the second part of the paper we consider sets which are on the borderline of being near a given orthonormal set.

The last part of this paper gives a simple formula for constructing sets near a given orthonormal set. As an application of this formula we obtain new properties of the so called non-harmonic Fourier series.

We shall handle these problems abstractly, using the notation of Hilbert space.[3] Subscript variables are assumed to range over all positive integers and $\sum$ shall mean a sum over all positive integers. By a finite sequence shall be meant a sequence with only a finite number of nonzero members. For application to the space L_2 the norm of a function $f(x)$ is defined in the usual way as $\|f\| = (\int_a^b |f(x)|^2 dx)^{1/2}$. A complete set which satisfies the Paley-Wiener criterion shall be termed strongly complete.

The principal novelty in the proof is the association of a linear transformation G with each set of elements $\{g_n\}$. Thus if $\{\psi_n\}$ is an orthonormal set we define $G\sum a_n\psi_n = \sum a_n g_n$ for every finite sequence of constants $\{a_n\}$. The norm of G is the limit superior of $\|Gx\|$ for elements x such that $\|x\| = 1$. With this definition of norm the aggregate of bounded linear transformations clearly forms a normed linear

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[1] R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, American Mathematical Society Colloquium Publications, vol. 19, 1934, p. 100.

[2] R. P. Boas, Jr., Journal of the London Mathematical Society, vol. 14 (1939), p. 242; Duke Mathematical Journal, vol. 6 (1940), p. 148; American Journal of Mathematics, vol. 63 (1941), p. 361.

[3] Because of the difficulty of finding adequate references to non-separable Hilbert space we confine ourselves to separable space. However, our theorems remain true for non-separable Hilbert space provided the range of subscript variable is redefined.