

## A NOTE ON THE LOWER SEMI-CONTINUITY OF DOUBLE INTEGRALS IN THE PARAMETRIC FORM

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Consider a double integral problem of the calculus of variations in parametric form, that is, a problem involving an integral of the form

$$\int \int_B f(x, X) du$$

where  $x$  stands for the three coordinate functions,  $x^1(u^1, u^2)$ ,  $x^2(u^1, u^2)$ ,  $x^3(u^1, u^2)$ , and  $X$  stands for their three Jacobians.

Various theorems of the following general type have been proved by McShane,<sup>1</sup> Caccioppoli,<sup>2</sup> Cimmino,<sup>3</sup> and Radó:<sup>4</sup>

If (i) the triples of functions  $x_n^i(u^1, u^2)$ ,  $i=1, 2$ ;  $n=0, 1, 2, \dots$ , satisfy certain conditions; (ii) the surfaces defined by  $x^i = x_n^i(u^1, u^2)$ ,  $i=1, 2$ ;  $n=1, 2, \dots$ , converge in some prescribed sense to the surface defined by  $x_0^i = x_0^i(u^1, u^2)$ ,  $i=1, 2$ ; (iii)  $f(x, X)$  satisfies certain conditions; then

$$\liminf_n \int \int_{B_n} f(x_n(u), X_n(u)) du \geq \int \int_{B_0} f(x_0(u), X_0(u)) du.$$

The conditions on  $f(x, X)$  usually contain a condition on its Weierstrass  $E$ -function

$$E(x, X, \bar{X}) = f(x, \bar{X}) - \bar{X}^\alpha f_\alpha(x, X),$$

the summation convention being used for convenience in writing.

McShane<sup>1</sup> requires either  $f > 0$ ,  $E \geq 0$ ; or  $f \geq 0$ ,  $E > 0$ . Caccioppoli<sup>2</sup> makes the same requirements. Cimmino<sup>3</sup> requires only  $f \geq 0$ ,  $E \geq 0$ , but works with smooth surfaces.

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<sup>1</sup> McShane, *On the semi-continuity of double integrals in the calculus of variations*, Annals of Mathematics, (2), vol. 33 (1932), pp. 460-484.

<sup>2</sup> Caccioppoli, *Gli integrali doppi di forma parametrica nel calcolo delle variazioni*, Arri del Reale Istituto Veneto di Scienze, Lettere ed Arti, Anno Accademico, 1933-1934, vol. 93, Part 2.

<sup>3</sup> Cimmino, *Sulle condizioni necessarie e sufficienti per la semi-continuità degli integrali doppi di forma parametrica*, Annali di Mathematica Pura ed Applicata, vol. 15 (1936), pp. 159-173. The author wishes to thank Professor McShane for calling his attention to this paper.

<sup>4</sup> Radó, *On the semi-continuity of double integrals in the parametric form*, Transactions of this Society, vol. 51 (1942), pp. 336-361. The author wishes to thank Professor Radó for access to the manuscript of this paper before it was published.