A NOTE ON THE DIOPHANTINE PROBLEM OF FINDING FOUR BIQUADRATES WHOSE SUM IS A BIQUADRATE

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The problem is to assign integers satisfying the relation

$$x^4 = y^4 + a^4 + b^4 + c^4. \tag{1}$$

Granted that the five integers have no common divisor, it follows that x and one quantity on the right (here taken as y) are not multiples of 5; and it can be shown that (1) without loss of generality can be replaced by

$$x^4 = y^4 + 5^4t^4(\alpha^4 + \beta^4 + \gamma^4). \tag{2}$$

In this expression x is an odd number prime to y; y may be odd or even; and neither is divisible by 5, as stated above.

Let $(x^4 - y^4)/5^4t^4 = d$; which for values satisfying (2) will be an integer.

Evidently x and y must satisfy the congruence $x^4 \equiv y^4 \pmod{5^4}$. Hence, for values of x up to any required magnitude, there are corresponding values of y;[1] and the resultant values of d can be found and tabulated.

If (2) is satisfied, then $d = \alpha^4 + \beta^4 + \gamma^4$; and from the elementary properties of the sum of three biquadrates it is seen that many values of d may be rejected at once: for example, all those having for the final digit 0, 4, 5, or 9; and all those incapable of representing the sum of three square numbers. The remaining values of d will have for the final digit 1, 2, 3, 6, 7, or 8. Regarding these as separate cases, it is possible to devise a numerical test for each case. As an example, consider the values of d ending with $\cdots 3$. These must have the form $80k+3$ (sum of three odd biquadrates prime to 5) or the form $80k+33$ (sum of one odd and two even biquadrates prime to 5). Choosing the latter form,[2] seventy instances are found in a tabulation of values of d based on all admissible values of $x < 700$. The first ten in order of magnitude are: 164833, 195313, 198593, 3029873, 4106193, 4590753, 5086913, 5693793, 5948193, 7424753. Testing these by a method (based upon the character of k), which need not be

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[1] Solutions of the congruence in question can be read from C. J. G. Jacobi's *Canon Arithmeticus*, Berlin, 1839, pp. 230–231.

[2] This form was considered first because it includes Norrie's solution, which appears in The University of St. Andrew 500th Anniversary Memorial Volume, Edinburgh, 1911, p. 89.