ON THE LEAST SOLUTION OF PELL'S EQUATION

LOO-KENG HUA

Let x_0, y_0 be the least positive solution of Pell's equation

$$x^2 - dy^2 = 4,$$

where d is a positive integer, not a square, congruent to 0 or 1 (mod 4). Let $\epsilon = (x_0 + d^{1/2}y_0)/2$. It was proved by Schur[1] that

$$\epsilon < d^{d^{1/2}}, \tag{1}$$

or, more precisely,

$$\log \epsilon < d^{1/2}((1/2) \log d + (1/2) \log \log d + 1) . \tag{2}$$

He deduced (1) from (2) by the property that

$$d^{1/2}((1/2) \log d + (1/2) \log \log d + 1) < d^{1/2} \log d$$

for $d > 244.69 \cdots$, and, for $d \leqq 244$, (1) is established by direct computation. It is the object of the present note to establish a slightly better result that

$$\log \epsilon < d^{1/2}((1/2) \log d + 1). \tag{3}$$

Thus (1) follows immediately without any calculation. The method used is that described in the preceding paper.

Let $(d|r)$ be Kronecker's symbol. (We extend the definition to include negative values of r by the relation $(d|r_1) = (d|r_2)$ for $r_1 \equiv r_2 \pmod{d}$.)

Let f denote the fundamental discriminant related to d, that is,

$$d = m^2 f,$$

where f is not divisible by a square of odd prime and is either odd, or congruent to 8 or congruent to 12 (mod 16).

LEMMA 1. *For $d > 0$, we have*

$$\left(\frac{d}{r}\right) = \left(\frac{d}{-r}\right).$$

PROOF. Landau, *Vorlesungen über Zahlentheorie*, vol. 1, Theorem 101.

LEMMA 2. *We have*

Received by the editors December 3, 1941.

[1] Göttingen Nachrichter, 1918, pp. 30–36.