ON PERRON INTEGRATION

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The definition of integral here presented had its origin in an unsuccessful attempt to establish the theorem on integration by parts[1] (Theorem 4.1 of this note) for the Perron integral, without detouring through the special Denjoy integral. In order to avoid the difficulties which I could not overcome, I amended the definition of the integral; the resulting definition I presented at the Oslo congress in 1936. Recently I have found a proof that the "new" integral is actually equivalent to that of Perron.

As compared with Perron's definition, the new definition has the slight disadvantage that it requires four associated functions instead of two; in all other respects the proofs of theorems for the Perron integral carry over unaltered. It has the advantage that it permits us to prove the general theorem on integration by parts (Theorem 4.1) without recourse to the deep-lying equivalence with the special Denjoy integral. This theorem is important, not only in itself, but because it also contains the corollary that if $f(x)$ is Perron integrable and $g(x)$ of limited total variation, their product is Perron integrable. Also it leads at once to the second theorem of the mean for Perron integrals.

1. Definition of the integral. Let $f(x)$ be a function defined on an interval $[a, b]$ and assuming values which are real numbers or $+\infty$ or $-\infty$. A set of four functions $\phi^i(x)$ $(i=1, 2, 3, 4)$ is a *tetrad adjoined to* $f(x)$ *on* $[a, b]$ if the following conditions are satisfied.

(1.1a) *The* $\phi^i(x)$ *are continuous on* $[a, b]$ *and all vanish at* $x=a$.

(1.1b) *For all except at most a denumerable collection of values of* x *the relations*

$$-\infty \neq D_+\phi^1, \qquad -\infty \neq D_-\phi^2,$$
$$+\infty \neq D^-\phi^3, \qquad +\infty \neq D^+\phi^4$$

are valid.

(1.1c) *For all except at most a denumerable collection of values of* x *the relations*

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[1] For a proof of the theorem with the added hypothesis that $g'(x)$ is finite except for a denumerable set, see R. L. Jeffery, *Perron integrals*, this Bulletin, vol. 48 (1942). pp. 714–717.