A NOTE ON COMPLEMENTATION IN LATTICES

L. R. WILCOX

The results of this note concern the notions of complementation and modularity in lattices. In the theories of modular lattices, for example, the theory of continuous geometries of von Neumann, the assumption of complementation has proved extremely powerful; the author has found that some analogue of this assumption is usually necessary in the study of non-modular lattices.[1] If L is a modular lattice with 0, 1, and if $a \in L$, then x is a complement of a in case $a+x=1$, $ax=0$. If L has the property

$$\text{(1)} \qquad \text{each } a \in L \text{ has a complement,}$$

then it may be proved that

$$\text{(2)} \qquad a, b \in L \text{ implies the existence of } b_1 \leqq b \text{ such that}$$
$$a + b_1 = a + b, \qquad ab_1 = 0.$$

Property (2) is the really useful one. Now if L is any lattice with 0, 1, then (1) does not imply (2); moreover, even (2) seems too weak for most purposes. However, if in (2) we assume in addition that (a, b_1) (or (b_1, a)) is a modular pair, then the usefulness of this assumption in analyzing the structure of L is considerably increased. At the same time, of course, limitations are placed on L, and it is these which we study here.

Let L be a lattice with 0 and 1. If $b, c \in L$, we say that (b, c) is a *modular pair* and write $(b, c)M$ if $(a+b)c=a+bc$ for $a \leqq c$. We write $(b, c) \perp$ if $bc=0$ and $(b, c)M$. The negation of M is M'. If $bc=0$ and $(b, c)M'$ we write $b \| c$. If $\perp$ is a symmetric relation, L is said to be $\perp$-*symmetric*; if M is symmetric, L is said to be M-*symmetric*.[2]

DEFINITION 1. *Let $a, b, b_1 \in L$. Then b_1 is a right complement within b of a in $a+b$ if*

$$b_1 \leqq b, \qquad a + b_1 = a + b, \qquad (a, b_1) \perp.$$

We say that b_1 is a left complement within b of a in $a+b$ if

$$b_1 \leqq b, \qquad a + b_1 = a + b, \qquad (b_1, a) \perp.$$

If $a \leqq b$, the phrase "within b" is omitted; if $a+b=1$, the phrase

Presented to the Society, September 5, 1941; received by the editors September 24, 1941.

[1] This Bulletin, abstract 47-5-208

[2] See L. R. Wilcox, *Modularity in the theory of lattices*, Annals of Mathematics, (2), vol. 40 (1939), pp. 490–505, for the elementary properties of the relations M and $\perp$.