A THEOREM ON LIE GROUPS

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It is the purpose of this note to prove the following theorem:

THEOREM 1. *Let G be a Lie group*[1] *and G^* a compact subgroup of G. Then there exists in G an open set O containing G^* with the property that for each subgroup H of G lying in O there is an element g of G such that $g^{-1}Hg$ is in G^*.*

Roughly, the theorem says that each subgroup near enough to G^* can be transformed into G^* by an appropriate element of G. This result can be regarded as a generalization of the known fact that Lie groups cannot have arbitrarily small subgroups (other than the identity), although it was not from this point of view that our interest arose. To make our meaning clear, assume that G^* is an invariant subgroup so that the factor group G/G^* is also a Lie group. If there were in G a subgroup H near G^* it would go, by the homomorphism taking G into G/G^*, into a subgroup near the identity of G/G^*. The only subgroup of G/G^* near the identity is the identity itself which means that if H is to be near G^* it must actually be a subgroup of G^*. We see that when G^* is an invariant compact subgroup of G, the conclusion of the theorem is true in a trivial sense.

Our proof of Theorem 1 in the more general situation is based on certain facts about the way in which G operates on the coset space G/G^* which will be denoted by M. This is the space whose points are the cosets gG^* of G^* in G. The group G acts transitively on M which can be regarded as a Riemannian space and Cartan[2] has shown that there exists in M a Riemannian metric for which G is a group of isometries. This fact will be of great importance in what follows.

We begin, as we may, by supposing that M is endowed with a Riemannian metric invariant under G and, furthermore, we assume that M has been made into a metric space (Fréchet) in the usual way

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[1] For all definitions and theorems on topological groups used in this paper see Pontrjagin, *Topological Groups*. The term compact, as used here, implies that the set is closed.

[2] *La Théorie des Groupes Finis et Continus et l'Analysis Situs*, Mémorial des Sciences Mathématiques, vol. 42, p. 43. For an excellent summary of properties of isometries and geodesics which will be useful here, see the paper by Myers and Steenrod, *The group of isometries of a Riemannian manifold*, Annals of Mathematics, (2), vol. 40 (1939), pp. 400–416.