ON LOCAL CONVEXITY IN HILBERT SPACE

I. J. SCHOENBERG

1. **Introduction.** Let M be a set of points in a normed vector space V. M is said to be convex if p and q being any two points of M, the whole rectilinear segment $\overline{pq}$ belongs to M. A weakened form of convexity, due to H. Tietze,[1] is as follows:

DEFINITION OF LOCAL CONVEXITY. *Let $p \in M$. The set M is said to be locally convex at p if there exists a positive $\rho = \rho(p)$ such that the intersection $M \cdot S(p; \rho)$, of M with the open sphere $S(p; \rho)$ of center p and radius ρ, is convex in the ordinary sense. The set M is called locally convex if M is locally convex at all its points.*

Every convex set is locally convex. The converse is not true since every open set is obviously locally convex. Tietze's chief result concerning local convexity is as follows:

THEOREM 1 (of Tietze). *Let E_k denote the k-dimensional euclidean space. A closed and connected set M in E_k which is locally convex is also convex in the ordinary sense.*

By means of his concept of local euclidean dimension of a set M at a point $p \in M$, Tietze reduces the proof of Theorem 1 to the case of locally convex continua with interior points and which coincide with the closure of their set of interior points. Tietze then proves the theorem for continua in E_2 and finally extends the proof to cover any E_k. Tietze's method does not seem to be applicable for sets in Hilbert space.

The following lines contain a simpler method of dealing with this problem which allows the establishment of Tietze's theorem in any real or complex normed vector space whether separable or not. For the sake of definiteness we state and prove our theorem for real Hilbert space.[2]

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[1] H. Tietze, *Über Konvexheit im kleinen und im grossen und über gewisse den Punkten einer Menge zugeordneten Dimensionszahlen*, Mathematische Zeitschrift, vol. 28 (1928), pp. 697–707.

[2] Recent results have demonstrated the breakdown of some classical properties of convex sets when we pass from euclidean spaces to Hilbert spaces. See, for example, David Moskowitz and L. L. Dines, *Convexity in linear spaces with an inner product*, Duke Mathematical Journal, vol. 5 (1939), pp. 520–534, in particular the example on