ON DIFFERENTIATION OF INTEGRALS AND APPROXIMATE CONTINUITY

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The following discussion is closely connected with Lebesgue's theorem that the derivative of an integral is equal to the integrand almost everywhere. It is well known that in generalizing this theorem to higher dimensions, great care must be exercised in the choice of the systems of intervals or sets used for n-dimensional differentiation.

Lebesgue[1] had already observed that arbitrary intervals (parallel to the axes) cannot be used for the generalization of that theorem, but only such intervals whose edges have a bounded ratio, or, more generally, such sets which are *regular* relative to the cubes. This also corresponds to the behavior of the most essential tool used in the proof, namely, Vitali's covering theorem. Saks[2] and, independently, Busemann and Feller[3] found later that there is a remarkable difference between the integrals of bounded[4] and unbounded functions: in the first (but not generally in the last) case, differentiation relative to arbitrary intervals (parallel to the axes) furnishes the integrand almost everywhere. But, according to Zygmund and Nikodym[5] and to Busemann and Feller,[3] even in the case of bounded integrands, differentiation relative to the system of all rectangular parallelopipeds (arbitrarily oriented) does not always furnish the integrand almost everywhere.

As to integrals in abstract spaces—in the case of bounded integrands, de Possel[6] gave necessary and sufficient conditions for the systems of sets used in differentiation to permit a generalization of Lebesgue's theorem; while in case of arbitrary integrands, de Possel[6]

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[1] H. Lebesgue, Annales de l'École Normale, (3), vol. 27 (1910), pp. 363, 387.

[2] S. Saks, *Théorie de l'Intégrale*, Warsaw, 1933, p. 232; *Theory of the Integral*, Warsaw, Lwow, 1937, p. 132.

[3] H. Busemann and W. Feller, Fundamenta Mathematicae, vol. 22 (1934), pp. 226–256.

[4] Further generalizations: A. Zygmund, Fundamenta Mathematicae, vol. 23 (1934), pp. 143–149; B. Jessen, J. Marcinkiewicz, A. Zygmund, Fundamenta Mathematicae, vol. 25 (1935), pp. 217–234.

[5] O. Nikodym, Fundamenta Mathematicae, vol. 10 (1927), pp. 167–168 (note cf. A. Zygmund).

[6] R. de Possel, Comptes Rendus de l'Académie des Sciences, Paris, vol. 201 (1935), pp. 579–581; Journal de Mathématiques, (9), vol. 15 (1936), pp. 391–409.