finally an analytic r-cell contained in $\mathfrak{g} \cap W$. Hence $\mathfrak{g}$ contains a nucleus of G and hence $\mathfrak{g} = G$, a contradiction which proves the theorem.[3]

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[3] We have proved, incidentally, that if an everywhere dense subgroup $\mathfrak{g}$ of a simple Lie group G_r $(r > 1)$ contains an analytic arc, then $\mathfrak{g} = G$.

VECTOR SPACES OVER RINGS

C. J. EVERETT[1]

1. **Introduction.** Let $\mathfrak{M} = u_1K + \cdots + u_mK$ be a vector space (linear form modul [5, p. 111]) over a ring $K = \{0, \alpha, \beta, \cdots; \epsilon \text{ unit element}\}$. By a *submodul* $\mathfrak{N} \leqq \mathfrak{M}$ is meant an "admissible" submodul: $\mathfrak{N}K \leqq \mathfrak{N}$. Elements $v_1, \cdots, v_n$ of a submodul $\mathfrak{N}$ form a *basis* for $\mathfrak{N}$ (notation: $\mathfrak{N} = v_1K + \cdots + v_nK$) in case $\sum v_i\alpha_i = 0$ implies $\alpha_i = 0$, $i = 1, \cdots, n$, and if every element of $\mathfrak{N}$ is expressible in the form $\sum v_i\alpha_i$, $\alpha_i \in K$. The equivalent formulations of the ascending chain condition for submoduls of a vector space, and for right ideals of a ring will be used without further comment [5, §§80, 97].

2. **Basis number, linear transformations.** We remark that the following holds.

(A) *The ascending chain condition is satisfied by the submoduls of a vector space $\mathfrak{M}$ over K if and only if it is satisfied by the right ideals of K.*

An infinite chain of right ideals $\mathfrak{r}_1 < \mathfrak{r}_2 < \cdots$ in K yields an infinite chain of submoduls $u_1\mathfrak{r}_1 < u_1\mathfrak{r}_2 < \cdots$ in $\mathfrak{M}$. The other implication is proved in [5, p. 87].

[By using a lemma due to N. Jacobson (*Theory of Rings*, in publication) Theorem (A) and the corresponding theorem for descending chain condition are easily proved in a unified manner.]

Linear transformations of $\mathfrak{M}$ on $\mathfrak{M}$ are given by $u_j \rightarrow u_j' = \sum u_i\alpha_{ij}$. Write $(u_1', \cdots, u_m') = (u_1, \cdots, u_m)A$, $A = (\alpha_{ij})$. Under $u_j \rightarrow u_j'$, let $\mathfrak{M}_0 \rightarrow 0$. Thus $\mathfrak{M}/\mathfrak{M}_0 \cong \mathfrak{M}A \leqq \mathfrak{M}$. Clearly $\mathfrak{M}_0 = 0$ if and only if $Av = 0$ implies $v = 0$, v an $m \times 1$ matrix over K, and $\mathfrak{M}A = \mathfrak{M}$ if and only if there exists an $m \times m$ matrix R with $AR = I$, the identity matrix.

Possibilities (i) $\mathfrak{M}_0 = 0$ and $\mathfrak{M}A = \mathfrak{M}$; (ii) $\mathfrak{M}_0 > 0$ and $\mathfrak{M}A < \mathfrak{M}$; (iii) $\mathfrak{M}_0 = 0$ and $\mathfrak{M}A < \mathfrak{M}$ are familiar. The possibility of (iv) $\mathfrak{M}_0 > 0$

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