EVERYWHERE DENSE SUBGROUPS OF LIE GROUPS

P. A. SMITH

A recent note by Montgomery and Zippin[1] leads one to speculate concerning the nature of everywhere dense proper subgroups of continuous groups. Such subgroups can easily be constructed. Suppose for example that G is a non-countable continuous group which admits a countable subset G_0 filling it densely. The group generated by G_0 is everywhere dense in G but is not identical with G. In the case of Lie groups, it is easy to see that an abelian G admits non-countable subgroups of the sort in question; whether or not a non-abelian G does so, appears to be a more difficult question. We shall, however, show that if G is simple, proper subgroups of G cannot, so to speak, fill G *too* densely.

Let G be a simple[2] Lie group of dimension r with $r>1$, and let U be a canonical nucleus of G—that is, a nucleus which can be covered by an analytic canonical coordinate system. An arbitrary point x of U is contained in the central of at least one closed proper Lie subgroup of G with non-discrete central. In fact, through x there passes a one-parameter subgroup γ; the closure of γ is an abelian Lie subgroup and this subgroup is proper since G is simple and $r>1$.

THEOREM. *Let G be a simple Lie group of dimension r greater than one and let $\mathfrak{g}$ be a proper subgroup filling G densely. There exists at least one proper closed Lie subgroup H of G such that those left- (right-) cosets of H which fail to meet $\mathfrak{g}$ fill G densely. For H one may take any closed proper Lie subgroup of G whose central is non-discrete and contains an arbitrarily chosen point p in $\mathfrak{g}\cap U$, U being any given canonical nucleus of G.*

PROOF. Let U, p, H be chosen and let us consider only the left-cosets of H. It will be sufficient to prove that there exists at least one coset, say aH, which fails to meet $\mathfrak{g}$. For, the cosets obtained by multiplying aH on the left by arbitrary elements of $\mathfrak{g}$ fail to meet $\mathfrak{g}$ and fill G densely.

Received by the editors July 15, 1941.

[1] Deane Montgomery and Leo Zippin, *A theorem on the rotation group of the 2-sphere*, this Bulletin, vol. 46 (1940), pp. 520–521. Our theorem may be regarded as a generalization of the theorem of Montgomery and Zippin and the proofs of the two theorems may be regarded as being the same in principle.

[2] We use *simple* here in the sense of having a simple Lie algebra. A simple group need not be connected.