ROOTS OF CERTAIN CLASSES OF POLYNOMIALS

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It is well known[1] that if the roots of the polynomials $\phi(z)$ and $F(z)$ are real, so are the roots of the polynomial $\phi(D)F(z)$, where $D=d/dz$. This result has been applied to certain types of entire functions and trigonometric integrals.[2] The following example illustrates the method employed. If

$$f(z) = \sum_{k=0}^{n} c_k z^k \tag{1}$$

is a polynomial whose roots $\lambda_1, \cdots, \lambda_n$ lie on the unit circle, then the roots of the polynomials

$$F_p(z) = c_n \prod_{k=1}^{n} [(1 + z/p)^p - \lambda_k(1 - z/p)^p], \qquad p = 1, 2, \cdots,$$

lie on the axis of pure imaginaries. Therefore, if the roots of the polynomial $\phi(z)$ lie on the axis of pure imaginaries, so do the roots of the polynomials[3] $\phi(D)F_p(z)$, $p=1, 2, \cdots$. Now the sequence of polynomials $\{F_p(z)\}$ converges uniformly in every finite region to the function

$$F(z) = e^{-nz}f(e^{2z}) = \sum_{k=0}^{n} c_k e^{(2k-n)z},$$

and the sequence $\{\phi(D)F_p(z)\}$ converges likewise to

$$\phi(D)F(z) = \sum_{k=0}^{n} c_k\phi(2k - n)e^{(2k-n)z}.$$

The roots of $\phi(D)F(z)$ therefore lie on the axis of pure imaginaries. Removing the innocuous factor e^{-nz}, and replacing e^{2z} by z, the following theorem results: *If the roots of $f(z)$ lie on the unit circle, and the roots of $\phi(z)$ lie on the axis of pure imaginaries, then the roots of the polynomial*

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[1] Ch. Hermite, Nouvelles Annales de Mathématiques, vol. 5 (1866), p. 479; Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis* II, p. 47, Problem 62.

[2] See G. Pólya, *Über trigonometrische Integrale mit nur reellen Nullstellen*, Journal für die reine und angewandte Mathematik, vol. 158 (1927), pp. 6–18.

[3] Replacing z by iz it follows from the above theorem of Hermite that if the roots of $\phi(z)$ and $F(z)$ lie on the axis of pure imaginaries, so do the roots of $\phi(D)F(z)$.