RESOLUTION OF BOUNDARY PROBLEMS BY THE USE OF A GENERALIZED CONVOLUTION

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1. The Laplace transformation of the convolution. The generalized convolution $F^*(t)$ of $F(t, t')$ is defined as follows:

$$F^*(t) = \int_0^t F(t - t', t')dt'.$$

In case $F(t, t') = F_1(t)F_2(t')$, the function $F^*(t)$ is the ordinary convolution $F_1 * F_2$, or Faltung,[1] of the two functions F_1 and F_2.

Let $L\{F^*(t)\}$ denote the Laplace transform of F^* with respect to t,

$$L\{F^*(t)\} = \int_0^\infty e^{-st}F^*(t)dt,$$

and let $\bar{f}(s)$ denote the iterated transform of $F(t, t')$,

$$\bar{f}(s) = \int_0^\infty e^{-st'}dt' \int_0^\infty e^{-st}F(t, t')dt. \tag{1}$$

It will be seen that

$$L\{F^*(t)\} = \bar{f}(s), \tag{2}$$

which, in terms of the inverse Laplace transformation, implies that

$$L^{-1}\{\bar{f}(s)\} = F^*(t).$$

THEOREM. *Let $F(t, t')$ be an integrable function of t and t' in every finite rectangle $0 \leqq t \leqq T$, $0 \leqq t' \leqq T'$ and, for some real α, let $e^{-\alpha(t+t')}|F(t, t')|$ be bounded for all t and t' ($t \geqq 0$, $t' \geqq 0$). Then if $R(s) > \alpha$, the integral $L\{F^*(t)\}$ is absolutely convergent and satisfies the equation* (2).

Under the conditions stated, the iterated integral in (1) exists if $R(s) > \alpha$ and is equal to the absolutely convergent double integral

$$\iint e^{-s(t+t')}F(t, t')d(t, t'),$$

over the quadrant $t \geqq 0$, $t' \geqq 0$. However, the latter is equal to

Presented to the Society, April 12, 1940, under the title *An extension of Duhamel's theorem*; received by the editors March 28, 1941.

[1] G. Doetsch, *Theorie und Andwendung der Laplace-Transformation*, Berlin, 1937, p. 155 ff.