

ARITHMETIC OF ORDINALS WITH APPLICATIONS TO THE THEORY OF ORDERED ABELIAN GROUPS

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1. **Introduction.**¹ The operations of addition and multiplication of ordinals do not behave as well as one might desire. For example, the commutative laws are not valid, and the distributive law is valid on only one side. Consequently, we make definitions of a sum and product that do not have such defects.

A binary operation, $\alpha \oplus \beta$, on ordinals is termed a *natural sum* if $\alpha \oplus \beta$ is a well-determined ordinal for any two ordinals, α and β , such that:

- (1) $\alpha \oplus \beta = \beta \oplus \alpha$,
- (2) $(\alpha \oplus \beta) \oplus \delta = \alpha \oplus (\beta \oplus \delta)$,
- (3) $\alpha \oplus 0 = \alpha$,
- (4) $\delta \oplus \alpha > \delta \oplus \beta$ if and only if $\alpha > \beta$,

where δ is any ordinal.

Throughout this paper, $\sigma(\alpha, \beta)$ will denote the natural sum defined by Hessenberg.² It is the unique natural sum satisfying the condition that $\omega^\alpha \cdot m + \omega^\beta \cdot n = \sigma(\omega^\alpha m, \omega^\beta n)$, where α and β are any two ordinals such that $\alpha \geq \beta$, and where m and n are any two positive integers. $\sigma(\alpha, \beta)$ shall be shown to be the "smallest" natural sum, and it shall be shown to be the best bound for the order type of the join of two well-ordered subsets, of respective order types α and β , of an ordered set.

A binary operation, $\alpha \otimes \beta$, on ordinals is termed a *natural product* if $\alpha \otimes \beta$ is a well-determined ordinal for any two ordinals, α and β , such that:

- (1) $\alpha \otimes \beta = \beta \otimes \alpha$,
- (2) $(\alpha \otimes \beta) \otimes \delta = \alpha \otimes (\beta \otimes \delta)$,
- (3) $\alpha \otimes 1 = \alpha$,
- (4) $\alpha \otimes \delta > \beta \otimes \delta$ if and only if $\alpha > \beta$,
- (5) $\sigma(\alpha \otimes \beta, \alpha \otimes \delta) = \alpha \otimes \sigma(\beta, \delta)$,
- (6) $\omega^\alpha \otimes \omega^\beta = \omega^\gamma$,

where δ is any ordinal, and where $\gamma = \gamma(\alpha, \beta)$ is a suitable ordinal.

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¹ The writer presupposes familiarity with the material on ordinals found in F. Hausdorff's *Mengenlehre*.

² G. Hessenberg, *Grundbegriffe der Mengenlehre*, Abhandlungen der Fries'schen Schule, (n. s.), 1.4, Göttingen, 1906, no. 75.