CONNECTED AND DISCONNECTED PLANE SETS AND THE FUNCTIONAL EQUATION $f(x)+f(y)=f(x+y)$

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Cauchy discovered before 1821 that a function satisfying the equation

$$f(x) + f(y) = f(x + y)$$

is either continuous or totally discontinuous.[1] After Hamel showed the existence of a discontinuous function satisfying the equation,[2] many mathematicians have concerned themselves with problems arising from the study of such functions.[3] However the following question seems to have gone unanswered: Since the plane image of such a function (the graph of $y=f(x)$) must either be connected or be totally disconnected, must the function be continuous if its image is connected? The answer is *no*.[4] The utility of this answer is at once apparent. For if $f(x)$ is totally discontinuous, its image obviously contains neither a continuum nor (in view of Darboux's work) a bounded connected subset even if the image itself is connected. As a matter of fact, if $f(x)$ is discontinuous but its image is connected, then the image, its complement, or some simple modification thereof, serves to illustrate rather easily many of the strange and non-intuitive properties of connected sets now illustrated by numerous complicated examples scattered through the literature. Thus this class of sets is a useful tool in studying connectedness and disconnectedness. A few illustrations are given, particularly in connection with

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[1] *Cours d'Analyse de l'École Royale Polytechnique*, part 1, *Analyse Algébrique*, 1921. This is Volume 3 of the 2d Series of Cauchy's *Complete Works* published by Gauthier-Villars et Fils, Paris, 1897, p. 99. Darboux in his paper, *Sur la composition des forces en statique*, Bulletin des Sciences Mathématiques, vol. 9 (1875), p. 281, showed (using Cauchy's methods) that if $f(x)$ is bounded in some interval, then $f(x)$ is continuous and of the form Ax.

[2] G. Hamel, *Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung*: $f(x+y)=f(x)+f(y)$, Mathematische Annalen, vol. 60 (1905), pp. 459–462.

[3] See in particular the early volumes of Fundamenta Mathematicae.

[4] It is odd that Sierpinski overlooked this, since about the time he published his papers on this subject he also published in Volume 1 of Fundamenta Mathematicae an example of a connected punctiform subset of the plane. And at this time he raised with Mazurkiewicz the question of the existence in the plane of a connected set containing no bounded connected subset.