SOME THEOREMS ON SUBSERIES

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1. **Absolutely convergent series.** A simple calculation reveals that the arithmetic mean value of all subsums (including the void sum) of a given finite sum $s_n = u_1 + u_2 + \cdots + u_n$ is equal to $s_n/2$. In this section we shall show (see Theorem 1 below) that an integral mean value can be found, consistent with the preceding, for the sums of all infinite subseries of a given absolutely convergent series $\sum u_k = s$. We begin by defining a one-to-one correspondence between the set of all infinite subseries of a given absolutely convergent real series $\sum u_k = s$, and the set of all points on the interval $I \equiv (0 < \xi \leqq 1)$. If ξ is any point of I then ξ admits a unique *nonterminating* binary representation of the form

$$\xi = 0.\alpha_1\alpha_2\alpha_3 \cdots \alpha_k \cdots \tag{1.1}$$

where

$$\alpha_{k_i} = 1 \quad (1 \leqq k_i < k_{i+1};\ i = 1, 2, 3, \cdots); \quad \alpha_k = 0 \quad \text{otherwise.} \tag{1.2}$$

To the point ξ shall correspond the infinite subseries $\sum_i u_{k_i}$. Conversely, if $\sum_i u_{k_i}$ $(1 \leqq k_i < k_{i+1})$ is a given infinite subseries of $\sum u_k$, we shall place it in correspondence with the point ξ of I defined by (1.1) and (1.2).

We now define a function $\phi(\xi)$ by setting $\phi(0) \equiv 0$ and

$$\phi(\xi) \equiv \sum_{k=1}^{\infty} \alpha_k u_k, \qquad 0 < \xi \leqq 1, \tag{1.3}$$

where $0.\alpha_1\alpha_2\alpha_3 \cdots \alpha_k \cdots$ is the nonterminating binary representation of ξ. In view of the above correspondence the set of all functional values $\phi(\xi)$ for ξ on I is evidently identical with the set of the sums of all infinite subseries of $\sum u_k$. This fact leads us to investigate the integrability of the function $\phi(\xi)$ and we find that the following lemma holds.

LEMMA 1. *The integral*

$$\int_0^1 \phi(\xi)d\xi \tag{1.4}$$

exists in the sense of Riemann, and has the value $s/2$.

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