CONFORMAL GEOMETRY OF ONE-PARAMETER FAMILIES OF CURVES

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A single regular analytic arc in the plane has no conformal differential invariants. The conformal theory of curvilinear angles was initiated by Kasner,[1] and has been elaborated by him and others. The present paper is concerned with conformal differential invariants of a real one-parameter family of regular analytic arcs in the plane. We assume that the family is defined in some region R of the (x, y)-plane by an equation of the form: $u(x, y)=\text{constant}$, where u is a single-valued function which satisfies the conditions: (1) u is analytic in the region R, (2) u assumes real values for real values of x and y, (3) $u_x^2+u_y^2$ does not vanish in R. By a conformal transformation we shall mean a real conformal transformation, nonsingular in R. Our principal results are: When a family $u=c$ is transformed conformally into a family $U=c$, the parameters of the two families being the same, the quantity $\Delta\equiv(u_{xx}+u_{yy})/(u_x^2+u_y^2)$, and certain conformally invariant derivatives of Δ are unaltered. There exist rational functions of Δ and these derivatives which are independent of the parameter in terms of which the family $u=\text{constant}$ is expressed. We obtain a geometric interpretation of the invariants and apply the results to a generalization of isothermal families.

1. **The invariants.** Let $U(X, Y)=c$ be a one-parameter family in the (X, Y)-plane. Let this plane be mapped conformally on the (x, y)-plane by the transformation

$$X = X(x, y), \qquad Y = Y(x, y), \tag{1.1}$$

where

$$X_x = Y_y, \quad X_y = -Y_x, \quad X_{xx} + X_{yy} = Y_{xx} + Y_{yy} \equiv 0. \tag{1.2}$$

The family $U(X, Y)=c$ is transformed into $u(x, y)=c$ where $u(x, y) \equiv U[X(x, y), Y(x, y)]$. By differentiating this last identity we obtain:

$$U_X X_x + U_Y Y_x = u_x, \qquad U_X X_y + U_Y Y_y = u_y. \tag{1.3}$$

These equations together with (1.2) give

$$u_x^2 + u_y^2 = J(U_X^2 + U_Y^2), \quad \text{where } J \equiv X_x^2 + X_y^2. \tag{1.4}$$

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[1] Proceedings of the International Congress at Cambridge, 1912.