PROJECTION OF THE SPACE (m) ON ITS SUBSPACE (c_0)

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In a paper in the Duke Journal, A. E. Taylor[1] remarks that it is an open question whether or not there exists a projection of the space (m), of bounded sequences, on its subspace (c_0), the space of sequences convergent to 0. In this note we make a few remarks which supplement those of Taylor on this question, and we point out that a negative answer follows from a recent result of R. S. Phillips,[2] so that the question is now settled.

Taylor shows that if a projection of the space (c), of convergent sequences, on the space (c_0) exists, it must be of norm greater than or equal to 2. This implies the same result for (m) on (c_0), since any projection of (m) on (c_0) would be in particular a projection of (c) on (c_0).

The space (c) is essentially of dimension only one greater than that of its subspace (c_0). This follows since (c) is obviously the set of all elements of the form $x=x^{(0)}+tX_1$, where $X_1=(1, 1, \cdots)$, $x^{(0)}\in(c_0)$, and t is a number. If $x=\{x_i\}$ is any element of (c), the linear functional $a(x)=t=\lim_{n\to\infty} x_n$ is of norm 1, and vanishes on the subspace (c_0). Now it is a remark of Bohnenblust[3] that for any subspace of a normed linear space L defined by the vanishing of a fixed linear functional on L, there exist projections of norm less than or equal to $2+\epsilon$, for arbitrary $\epsilon>0$. Consequently there are projections of (c) on (c_0) of norm less than or equal to $2+\epsilon$.

There are projections of (c) on (c_0) which are of norm exactly 2, as may be seen as follows. If $x=(x^{(0)}+tX_1)\in(c)$, the general form of a projection of (c) on (c_0) is

$$Px = x + t\{b_i\} = x^{(0)} + t(X_1 + \{b_i\})$$

where $\{b_i\}$ is any sequence of constants such that $\lim_{i\to\infty} b_i=-1$.[4] To calculate the norm of P, we have $\|Px\|\leqq\|x\|+|t|\cdot\sup_i |b_i|$, and $\|x\|=\|x^{(0)}+tX_1\|=\sup_i |x_i^{(0)}+t|\geqq|t|$ since $x_i^{(0)}\to 0$. Therefore $|P|\leqq 1+\sup_i |b_i|$, and because of Taylor's result this has the value

[1] *The extension of linear functionals*, Duke Mathematical Journal, vol. 5 (1939), pp. 538–547; p. 547.

[2] *On linear transformations*, Transactions of this Society, vol. 48 (1940), pp. 516–541; pp. 539–540.

[3] *Convex regions and projections in Minkowski spaces*, Annals of Mathematics, (2), vol. 39 (1938), pp. 301–308; p. 308.

[4] See Taylor, op. cit.