A THEOREM CONCERNING THE GEODESICS ON A PARABOLOID OF REVOLUTION[1]

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We shall examine the behavior in the large of the geodesics on a paraboloid of revolution. It will appear that on every geodesic there is a unique point—the "point of symmetry"—which divides the geodesic into two "conjugate rays," which are mirror images of each other in the plane determined by that point and the axis of the paraboloid. Every geodesic except the meridians, which have no singularities whatever, has infinitely many double points, but no other singular points. If, starting at the point of symmetry, we imagine the conjugate rays of a geodesic to be traversed simultaneously by two points which are images in the plane of symmetry, and which continue until they meet for the first time—at the "first double point" of the geodesic—we shall have generated a "loop of type 1." If, starting at the point of symmetry, they pass through the first double point and continue until they next meet—at the "second double point"—we shall have formed a "loop of type 2." We define in this way loops of types 1, 2, 3, $\cdots$. The last double point to be reached in the process of generating a loop of a given type will be called the "vertex" of the loop.

The main result of this paper is the following: The paraboloid can be divided by planes perpendicular to the axis into infinitely many zones, each zone to include the points of that bounding circle which is nearer the vertex. The zone containing the vertex we call the "cap," the following the "first zone," and so on. The zones have the following properties:

1. Each interior point of the kth zone is the vertex of exactly two distinct loops of each of the types 1, 2, $\cdots$, k, but is the vertex of no loop of type greater than k.

2. Each boundary point in the kth zone is the vertex of two loops of each of the types 1, 2, $\cdots$, $k-1$ and of one loop of type k. It is the vertex of no loop of type greater than k.

3. No point of the cap is the vertex of any loop.

Let the paraboloid be that obtained by revolving $r^2=x$ about the x-axis. As coordinates of a point P on the surface (the vertex excepted) we take r to be the length of the vector R from P perpendicular to the axis and directed toward P, and θ to be the angle which the plane determined by P and the axis makes with an arbitrary plane contain-

[1] Presented to the Society, February 24, 1940.