A PROBLEM IN PARTITIONS[1]

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Let m objects $x_1, \cdots, x_m$ be given and from these n non-void subsets $a_1, \cdots, a_n$ be formed. This partition will determine a matrix (a_{ij}) in which $a_{ij}=1$ if a_i and a_j have a non-void intersection and $a_{ij}=0$ if a_i and a_j are disjoint. Necessarily (a_{ij}) is a symmetric matrix with 1's on the main diagonal. The following question has arisen in Ore's investigation of the theory of relations: Is every matrix (a_{ij}), $i, j=1, \cdots, n$, with $a_{ii}=1$, $a_{ij}=a_{ji}=0$ or 1 the partition matrix of n objects into n non-void subsets? As will be seen presently, the answer to this question is in the negative. The reason is not that there is any inherent contradiction within certain matrices but that it is not always possible to find a partition of as few as n objects determining a given matrix.

The answer is affirmative for $n=1, 2, 3, 4$ as may be found by direct calculation, but is negative for $n \geqq 5$. It is almost trivial that for $n \geqq 3$, $m=(n^2-n)/2$ objects will suffice. Take $(n^2-n)/2$ objects $u_{ij}=u_{ji}$, $i \neq j$, $i, j=1, \cdots, n$, and assign u_{ij} to both a_i and a_j if $a_{ij}=1$ and discard u_{ij} if $a_{ij}=0$. This will leave certain subsets a_i void for which $a_{ii}=1$, $a_{ij}=0$ if $j \neq i$, and for these we introduce new objects u_i in a_i alone. If there are one or two such i's we have discarded at least two u_{ij}'s since $n \geqq 3$. If there are $s \geqq 3$ such i's we have discarded at least $s(s-1)/2$ u_{ij}'s, namely those with both subscripts from this set. In all events we have discarded at least as many objects as we have added and we have a partition of $(n^2-n)/2$ or fewer objects into n non-void subsets corresponding to the prescribed partition matrix. But it is clear that this number $m=(n^2-n)/2$ is too high for $n>3$ since the full number of objects is used only if every $a_{ij}=1$ and in this case a single object assigned to every subset will suffice.

THEOREM 1. *A given matrix (a_{ij}), $i, j=1, \cdots, n$, in which $a_{ii}=1$, $a_{ij}=a_{ji}=0$ or 1, is the partition matrix of a set of at most n objects for $n=1, 2, 3, 4$ and of at most $n^2/4$ (n even and $n \geqq 4$) or $(n^2-1)/4$ (n odd and $n \geqq 5$).*

PROOF. Evidently renumbering the subsets makes no difference in the problem. This operation corresponds to permuting both the rows and columns of the matrix, the same permutation being applied to both. Such an operation defines an equivalence on the matrices. The

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