ON THE EXISTENCE OF ELECTRICAL NETWORKS

C. E. CLARK

This paper is concerned with the existence of electrical networks which satisfy certain preassigned conditions. These conditions have to do with the existence of circuits with preassigned resistances in common.

Consider a finite set of points in euclidean 3-space and a set of straight line segments joining pairs of these points. Furthermore, suppose that no two of the segments intersect at an interior point, but any number of segments may have a common end-point. Each of the segments is called a *branch* of the graph. With each branch of the graph let there be associated a non-negative real number called the *resistance* of the branch. The graph together with the resistances is an *electrical network*. A *circuit* of a network is a topological circle of the network together with an orientation of this circle.

Let two circuits C_i and C_j have the branches b_{ij}^p, $p=1, 2, \cdots$, in common. Let r_{ij}^p be the resistance of b_{ij}^p. Let $\bar{r}_{ij}^p = r_{ij}^p$ if the orientations of C_i and C_j agree along b_{ij}^p while $\bar{r}_{ij}^p = -r_{ij}^p$ if the orientations are opposite. Then $I_{ij} = \sum_p \bar{r}_{ij}^p$, $p=1, 2, \cdots$, is the *intersection* of C_i and C_j. We see that I_{ii} is the sum of the resistances of the branches of C_i.

Let C_i, $i=1, 2, \cdots, n$, be distinct circuits of a network. Then the matrix $\|I_{ij}\|$, $i, j=1, \cdots, n$, is the *intersection matrix of the* C_i. This matrix is symmetric and has non-negative diagonal elements. Any matrix with these two properties is an *intersection matrix*. An intersection matrix M is *realizable* when there exists a network which has a set of circuits whose intersection matrix is M.

THEOREM 1. *Given an intersection matrix*

$$\|I_{ij}\|, \qquad i, j = 1, \cdots, n; \tag{1}$$

if $I_{ii} \geqq \sum_j |I_{ij}|$, $j \neq i$, $i=1, \cdots, n$, *then* (1) *is realizable.*

PROOF. Choose $2n$ points on each of n oriented circles S_i, $i=1, \cdots, n$. These points divide each S_i into $2n$ branches. Denote the $2n$ branches of S_i by $b_{i1}, o_{i1}, b_{i2}, o_{i2}, \cdots, b_{in}, o_{in}$ with the b's separated by the o's. To each o is assigned the resistance zero. We shall next identify b_{ij} and b_{ji}, $i \neq j$.[1] In the resulting figure there will be a circle that is the natural topological image of S_i. We shall say that this circle *is* S_i.

[1] I.e., we bring the two end-points of b_{ij} into coincidence with the end-points of b_{ji} and replace these two branches by a single branch. To do this it may be necessary to replace any figure by its homeomorph.