# A NOTE ON FUNCTIONS OF EXPONENTIAL TYPE[1]

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An entire function $f(z)$ is said to be of exponential type at most $T$ if

$$\limsup_{n\to\infty} |f^{(n)}(z)|^{1/n} \leqq T \tag{1}$$

for some $z$ (and hence for every $z$, uniformly for $z$ in any bounded set). An equivalent condition[2] is that for each positive $\epsilon$

$$|f(z)| < e^{(T+\epsilon)|z|}$$

for all sufficiently large $|z|$. The following three theorems were proved respectively by D. V. Widder [**4**], I. J. Schoenberg [**2**], and H. Poritsky [**1**] and J. M. Whittaker [**3**].

THEOREM 1. (Widder.) *If a real function* $f(z)$, *of class* $C^\infty$ *in* $0 \leqq x \leqq 1$, *satisfies the condition*

$$(-1)^n f^{(2n)}(x) \geqq 0, \qquad 0 \leqq x \leqq 1;\ n = 0, 1, 2, \cdots, \tag{2}$$

*then* $f(x)$ *coincides on* $(0, 1)$ *with an entire function of exponential type at most* $\pi$.

THEOREM 2. (Schoenberg.) *If* $f(z)$ *is an entire function of exponential type at most* $T$, *and if*

$$f^{(2n)}(0) = f^{(2n)}(1) = 0, \qquad n = 0, 1, 2, \cdots, \tag{3}$$

*then* $f(z)$ *is a sine polynomial of order at most* $T/\pi$:

$$f(z) = \sum_{k=0}^{N} a_k \sin k\pi z, \qquad N \leqq T/\pi.$$

Let $\Lambda_n(z)$ be the polynomial of degree $2n+1$ determined by the relations

$$\Lambda_0(z) = z; \qquad \Lambda_n(0) = \Lambda_n(1) = 0, \qquad n \geqq 1;$$

$$\Lambda_n''(z) = \Lambda_{n-1}(z), \qquad n \geqq 1.$$

THEOREM 3. (Poritsky-Whittaker.) *If* $f(z)$ *is an entire function of exponential type at most* $T$, $T < \pi$, *then* $f(z)$ *can be represented in the form*

[1] Presented to the Society, February 22, 1941.

[2] G. Valiron, *Lectures on the General Theory of Integral Functions*, Toulouse, 1923, p. 41.