ON A GENERALIZED GREEN'S FUNCTION AND CERTAIN OF ITS APPLICATIONS[1]

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1. **Introduction.** A theorem of Blaschke in the theory of a.f. 1 c.v. (analytic functions of one complex variable) states: $\sum_{\nu=1}^{\infty}\log|a_\nu| > -\infty$, $|a_\nu| < 1$, is a necessary and sufficient condition for the existence of a non-negative, harmonic function[2] $H(z)$, $z \in [\mathfrak{G}^2 - \mathfrak{S}_{\nu=1}^{\infty}\{a_\nu\}]$, $\mathfrak{G}^2 = \mathcal{E}[|z| < 1]$, which possesses the property that $[H(z) + \log|z - a_\nu|]$, $\nu = 1, 2, \cdots$, is regular in a neighborhood of $z = a_\nu$. By

$$\exp[-H(z) - iB(z)] = f(z)$$

where $B(z)$ is a function conjugate to $H(z)$ we obtain a function $f(z)$, $|f(z)| \leqq 1$, $z \in \mathfrak{G}^2$, which possesses factors $(z - a_\nu)$, $\nu = 1, 2, \cdots$.

If one wishes to obtain an analogous result in the theory of a.f. 2 c.v. one must bear in mind at first the following fact:

If $f(z)/g(z)$ is regular in $\mathfrak{G}^2$ we call $g(z)$ a zero function of f in $\mathfrak{G}^2$.

Since every function f regular in $\mathfrak{G}^2$ can be represented in the form $f(z) = \prod(z - a_\nu)k_\nu(z)$ where $k_\nu(z)$ are regular and nonvanishing in $\mathfrak{G}^2$, we need to consider in the theory of a.f. 1 c.v. only *linear zero functions*. In the case of a.f. 2 c.v. we cannot in general represent even polynomials as products of linear functions; therefore, one must use for *zero functions* not only linear expressions but also arbitrary a.f. 2 c.v. [1, p. 1189].[3]

Furthermore there is lacking in the theory of a.f. 2 c.v. a theorem analogous to the theorem of Riemann, stating that every simply connected domain possessing at least two boundary points can be transformed conformally into $\mathfrak{G}^2$. We cannot therefore limit ourselves to

[1] Presented to the Society, April 27, 1940.

[2] We designate by capital and small letters, respectively, real and complex functions of z_k, $z_k = x_k + iy_k$, and manifolds by English letters, where the upper index denotes the dimension of the manifold. We omit this index for four-dimensional manifolds. We denote by $\mathcal{E}[\cdots]$ the set of points whose coordinates satisfy the relations indicated in brackets. $\mathfrak{S}$ means the logical sum. A horizontal bar above a letter indicates the closure of the set denoted by the letter.

[3] The numbers in brackets refer to the following papers: Stefan Bergman, **1.** Proceedings, Akademie van Wetenschappen, Amsterdam, vol. 34 (1932), pp. 1188–1194, **2.** Mathematische Annalen, vol. 102 (1934), pp. 324–348, **3.** Compositio Mathematica, vol. 3 (1936), pp. 136–173, **4.** Compositio Mathematica, vol. 6 (1939), pp. 305–335, **5.** Stefan Bergman and Marcinkiewicz, Fundamenta Mathematicae, vol. 33 (1939), pp. 75–94, **6.** G. Buler, Bulletin de l'Institute Mathématique de Tomsk, vol. 2 (1939), pp. 164–186, **7.** S. Saks, *Theory of the Integral*, Monografie Matematyczne, vol. 7, Warsaw and Lwów, 1937.