AN INTEGRAL ANALOGUE OF LAPLACE'S EQUATION[1]

MAXWELL READE AND E. F. BECKENBACH

1. **Introduction.** A general problem of a formal nature may be stated, for functions of two variables, as follows. Let a class of functions $x(u, v)$, or a class of sets of functions $x_j(u, v)$, $j=1, 2, \cdots, n$, defined in a finite simply connected domain D, be characterized as satisfying a set of one or more differential equations,

$$A = 0. \tag{1}$$

If in (1) we replace the partial derivatives

$$\frac{\partial y}{\partial u}, \quad \frac{\partial y}{\partial v}$$

by the line integrals

$$\frac{1}{\pi r^2}\int_{C_r} y dv, \qquad -\frac{1}{\pi r^2}\int_{C_r} y du,$$

respectively, where C_r is an arbitrary circle, of variable radius r, lying in D, we obtain a set of integral equations,

$$B = 0, \tag{2}$$

analogous to (1). Our problem is the determination of the class of functions, or of sets of functions, characterized by (2).

If $y(u, v)$ has continuous first partial derivatives, then

$$\frac{1}{\pi r^2}\int_{C_r} y dv = y_u(u_0, v_0) + o(r^0), \qquad -\frac{1}{\pi r^2}\int_{C_r} y du = y_v(u_0, v_0) + o(r^0),$$

where the subscripts u, v denote partial differentiation, where r and (u_0, v_0) are the length of the radius and the coordinates of the center of C_r, respectively, and where $o(r^\alpha)$ denotes a quantity (not always the same quantity) such that

$$\lim_{r\to 0} \frac{o(r^\alpha)}{r^\alpha} = 0.$$

Hence, for functions for which the derivatives involved in (1) are continuous, (1) is equivalent to

$$B = o(r^0),$$

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