THE ROOTS OF A POLYNOMIAL AND ITS DERIVATIVE[1]

JOHN C. MONTGOMERY

Consider a polynomial $f(z)$ of degree n, with the roots $\alpha_1, \cdots, \alpha_n$ and its first derivative $f'(z)$ with roots $\beta_1, \cdots, \beta_{n-1}$. The classical problem, where certain of the roots α_i are given and conclusions are drawn concerning the location of the roots β_i, will be considered in the converse. By means of an identity between these two sets of roots, two theorems will be given which restrict the location of the roots of the polynomial $f(z)$ when certain of the roots of its derivative, $f'(z)$ are equally placed on the unit circle.

THEOREM 1. *If $\alpha_1, \cdots, \alpha_n$, the n roots of the polynomial $f(z)$ of degree n, are different from the q distinct roots $\beta_1, \cdots, \beta_q$, $q \leqq n-1$, of the derived polynomial $f'(z)$, then*

$$\sum_{j=1}^{n} \frac{1}{(\beta_1 - \alpha_j)(\beta_2 - \alpha_j) \cdots (\beta_q - \alpha_j)} = 0. \tag{1}$$

The expansion of $1/(\beta_1 - x) \cdots (\beta_q - x)$ into partial fractions when summed as x ranges over the values $\alpha_1, \cdots, \alpha_n$, yields

$$\sum_x \frac{1}{(\beta_1 - x) \cdots (\beta_q - x)} = \sum_x \frac{A_1}{(\beta_1 - x)} + \cdots + \sum_x \frac{A_q}{(\beta_q - x)},$$

where the coefficients $A_1, \cdots, A_q$ are expressions of $\beta_1, \cdots, \beta_q$, and thus may be taken outside of the summations. Then the theorem follows since the remaining sums are all identically zero by the well known expression

$$\frac{f'(\beta_i)}{f(\beta_i)} = \sum_{j=1}^{n} \frac{1}{(\beta_i - \alpha_j)} = 0, \qquad i = 1, 2, \cdots, q.$$

In a recent paper by M. Marden[2] an identity is given between q roots of $f'(z)$ and p roots of $f(z)$ of degree n, $n = p + q - 1$. This identity gave a clue to the relation (1). In fact (1) may be derived by means of a complete induction operating on the Marden identity.

In order to study the effect that (1) has on the distribution of the roots of the polynomial $f(z)$, we introduce the following notation. Let

[1] Taken from a dissertation presented for the degree of Doctor of Philosophy in Yale University.

[2] M. Marden, *Kakeya's problem*, Transactions of this Society, vol. 45 (1939), pp. 355–368.