LINEAR FUNCTIONALS AND INTEGRALS IN ABSTRACT SPACES[1]

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In his addendum to Saks, *Theory of the Integral*, Banach[2] considers a Lebesgue integral defined in a manner quite similar to that of Daniell[3] and remarks that no use is made of a measure. It is, however, quite easy to show that Banach's and Daniell's integrals are expressible as Lebesgue integrals whose measure functions are regular outer measures in the sense of Carathéodory.

In the first two sections a linear, non-negative functional is considered. Upper and lower functionals are associated with this functional, and by means of them inner and outer measures are defined. It is shown that if the inner and outer measures of a set coincide, the set is measurable. To establish the converse a continuity assumption is made in §3, and a representation theorem in terms of the Lebesgue integral is obtained. It is shown in §4 that the theorem of Lebesgue for term-wise integration holds for semi-uniformly convergent $\mathfrak{R}R$-systems.

1. **Preliminary definitions.** It will be convenient to consider two linear classes of real-valued functions defined on a completely arbitrary range $\mathfrak{P}$. The first of these sets will be symbolized by $\mathfrak{X}$ and it will be supposed to contain the absolute value of every function in it. The second set $\mathfrak{Z}$ is made up of all functions $z(p)$ such that for some x in $\mathfrak{X}$, $|z| \leqq x$. In accordance with Daniell's notation the symbols $x_1 \vee x_2$, $x_1 \wedge x_2$ represent the larger and smaller, respectively, of the functions x_1, x_2 at each place p. Both these functions are in $\mathfrak{X}$ since $x_1 \vee x_2 = x_2 + (x_1 - x_2) \vee 0$, $x_1 \wedge x_2 = x_1 - (x_1 - x_2) \vee 0$ and since $x \vee 0 = (x + |x|)/2$.

Throughout this paper we shall be concerned with a linear functional I which is defined on the class $\mathfrak{X}$ and is non-negative in the sense that $x \geqq 0$ implies $I(x) \geqq 0$. Associated with this functional I there are two others I^*, I_* which will be called the upper and lower functionals and are defined by the equations

$$I^*(z) = \underset{z \leqq x}{\text{g.l.b.}}\, I(x), \quad I_*(z) = \underset{z \geqq x}{\text{l.u.b.}}\, I(x).$$

[1] Presented to the Society, April 13, 1940.

[2] Op. cit., p. 320 ff.

[3] *A general form of integral*, Annals of Mathematics, vol. 19 (1918), pp. 279–294.