BICOMPACTNESS OF CARTESIAN PRODUCTS

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1. **Introduction.** Tychonoff [7] was the first to prove that the cartesian product of any number of bicompact spaces is bicompact. Of the other proofs[1] in the literature [2, 6] perhaps the simplest is that of Tukey, which involves the notion of an *ultraphalanx*. In the present note a proof of a rather general form of this theorem is given, using only simple machinery. It is shown that the same method can be used to prove that the cartesian product of any number of absolutely closed Hausdorff spaces is an absolutely closed Hausdorff space.

2. **Definitions.** The spaces considered are those in which an operation of *closure* $\bar{A}$ of a set A is defined in terms of *neighborhoods* in the usual way, that is, x is a point of $\bar{A}$ if and only if every neighborhood of x contains a point of A. It follows that the closure operation is *monotone*; in other words, $\bar{A} \subset \bar{B}$ if $A \subset B$. Conversely, any closure operation which is monotone can be defined in terms of neighborhoods. No assumptions are made about the neighborhoods of a point, except that when they exist, they are sets of points.

The *cartesian product* P of a collection of such neighborhood spaces $\{B_k\}$ is a space whose points p are all selections $\{p_k\}$ containing just one point p_k from each of the spaces B_k. Neighborhoods are defined in P as follows. To any neighborhood N_k in B_k of a coordinate p_k of p, there corresponds in the product space P the neighborhood W_k of p consisting of all points q of P whose coordinate q_k is in N_k. The intersection of any finite collection $\{W_{k_r}\}$, $r=1, \cdots, n$, of neighborhoods of p of this type, such that no two subscripts k_r are the same, is also defined to be a neighborhood of p. This is the usual definition of cartesian product due to Tychonoff [7]. Note that it is not true in general that the intersection of any finite number of neighborhoods of p is a neighborhood of p.

A system $\mathfrak{S}$ of sets is said to have the *finite intersection property* if every finite number of sets of $\mathfrak{S}$ has at least one common point. It can be shown by a familiar argument, using Zorn's lemma or transfinite induction [4, 6, 8], that any system $\mathfrak{S}$ of subsets of a given set

[1] See also J. W. Alexander, *Ordered sets, complexes, and the problem of compactification*, Proceedings of the National Academy of Sciences, U.S.A., vol. 25 (1939), pp. 296–298, and E. Čech, *On bicompact spaces*, Annals of Mathematics, (2), vol. 38 (1937), p. 830.