VALUE REGIONS FOR CONTINUED FRACTIONS[1]

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1. **Introduction.** In a recent paper[2] we showed that if $a_2, a_3, a_4, \cdots$ lie in or upon the parabola

$$|z| - \Re(z) = \tfrac{1}{2} \tag{1.1}$$

then the continued fraction

$$\frac{1}{1} + \frac{a_2}{1} + \frac{a_3}{1} + \frac{a_4}{1} + \cdots \tag{1.2}$$

converges if and only if the series $\sum |b_n|$ diverges, where $b_1 = 1$, $a_n = 1/b_{n-1}b_n$, $(n = 2, 3, 4, \cdots)$, the series being considered as divergent if some a_n vanishes. Further, if z lies outside this parabola, the periodic continued fraction

$$\frac{1}{1} + \frac{\bar{z}}{1} + \frac{z}{1} + \frac{\bar{z}}{1} + \frac{z}{1} + \cdots, \tag{1.3}$$

in which $\bar{z}$ is the conjugate of z, diverges, so that the parabola (1.1) is the "best" curve symmetrical with respect to the real axis.

The principal object of the present paper is to show that when $a_2, a_3, a_4, \cdots$ lie in or upon the parabola (1.1) then all the approximants of (1.2) lie in or upon the circle

$$|z - 1| = 1. \tag{1.4}$$

If z_1 is any value of z not zero which is in or upon this circle, then there is a value z in or upon the parabola (1.1) such that the value of the continued fraction (1.3) is z_1, and therefore the circular domain is the "best" domain.

2. **Fundamental lemma.** If we adopt the notation

$$v_n = \frac{1}{1} + \frac{a_{n+1}}{1} + \frac{a_{n+2}}{1} + \cdots,$$

then

$$v_n = \frac{1}{1 + a_{n+1}v_{n+1}}, \qquad n = 1, 2, 3, \cdots.$$

[1] Presented to the Society under the title *A geometrical method in the theory of continued fractions*, November 22, 1940.

[2] W. T. Scott and H. S. Wall, *A converge theorem for continued fractions*, Transactions of this Society, vol. 47 (1940), pp. 155–172, p. 166. We refer to this paper later as CT.