INEQUALITIES OF A. MARKOFF AND S. BERNSTEIN FOR POLYNOMIALS AND RELATED FUNCTIONS[1]

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Introduction. Weierstrass was the first to prove that an arbitrary continuous function which is defined over a closed finite interval may be uniformly approximated by a sequence of polynomials. The more difficult problem of best approximation by polynomials had earlier been initiated by Tchebycheff. A number of years later, in the early part of the present century, de la Vallée Poussin raised the following question of best approximation: Is it possible to approximate every polygonal line by polynomials of degree n with an error of $o(1/n)$ as n becomes large? (He had proved that the approximation can be carried out with an error of $O(1/n)$. This question was answered in the negative by Serge Bernstein in a prize-winning essay on problems of best approximation. In this paper Bernstein proved and made considerable use of an inequality concerning the derivatives of polynomials. This inequality and a related (and earlier) one by Andrew Markoff have been the starting point of a considerable literature. It has been found for example that the underlying ideas of these two inequalities are applicable to a much wider class of functions than polynomials. These inequalities have supplied one approach to questions concerning the derivatives of quasi-analytic functions. A generalization of Bernstein's theorem has been applied to almost periodic functions.

In discussing a mathematical theory we may emphasize either its applications or the salient points of the theory itself. The applications of Bernstein's inequality to problems of approximation (where it has probably had its greatest success) have been treated in the literature; see for example Dunham Jackson's book in the Colloquium Publications of the American Mathematical Society. On the other hand I am unaware of any recent résumé of the literature which has been suggested by the theorems of Markoff and Bernstein, so I shall discuss some of the investigations which have centered about these theorems.

Rational polynomials. It was the chemist Mendeleieff (author of the periodic table of chemistry) who asked the following question: If the bound of a rational polynomial over a given interval is known, how large may its derivative be in this interval? The maximum possible value of the derivative will of course depend on the degree of the poly-

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