# ON THE SIMULTANEOUS APPROXIMATION OF TWO REAL NUMBERS[1]

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If $\xi_1, \xi_2, \cdots, \xi_n$ are any real numbers and $t$ is a positive integer, then it is well known that integers $a_1, a_2, \cdots, a_n, b$ can be found, such that $0 < b \leqq t^n$ and

$$|b\xi_k - a_k| < 1/t, \qquad k = 1, 2, \cdots, n.$$

The proof is briefly the following.[2] Consider the $t^n+1$ points $(r\xi_1, r\xi_2, \cdots, r\xi_n)$, where $r=0, 1, \cdots, t^n$. Reduce mod 1 to congruent points in the unit cube $(0 \leqq x_1 < 1, \cdots, 0 \leqq x_n < 1)$. If this cube is divided into $t^n$ cubes of edge $1/t$ (including the lower boundaries), then at least one of these small cubes must contain two of the reduced points, say those with $r=r'$ and $r=r''$. With $b=|r'-r''|$ and suitable $a$'s, we evidently satisfy the required inequalities.

For $n=1$, the inequality can be sharpened to

$$|b\xi - a| \leqq 1/(t+1),$$

$b$ satisfying the condition $0 < b \leqq t$. For if we consider the points $r\xi$ $(r=0, 1, \cdots, t)$, and mark the points in the interval $0 \leqq x \leqq 1$ which are congruent to them mod 1, we have at least $t+2$ points marked, since corresponding to $r=0$ we mark both 0 and 1. Some two of the marked points must lie within a distance $1/(t+1)$ from each other, so that the desired conclusion follows. This is the best result, as the example $\xi = 1/(t+1)$ shows.

The present note solves the corresponding problem for $n=2$. For larger values of $n$ the problem appears more difficult.

THEOREM. *If $\xi_1$ and $\xi_2$ are any real numbers, and $s$ is a positive integer, then integers $a_1, a_2, b$ can be found, such that $0 < b \leqq s$, and*

$$|b\xi_k - a_k| \leqq \max\left(\frac{[s^{1/2}]}{s+1}, \frac{1}{[s^{1/2}]+1}\right), \qquad k = 1, 2.$$

*For every $s$, values of $\xi_1$ and $\xi_2$ can be found for which the inequalities could not both be satisfied if the equality sign were omitted.*

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[1] Presented to the Society, November 23, 1940.

[2] The method used in this proof (*Schubfachprinzip* or "pigeonhole principle") was first used by Dirichlet in connection with a similar problem. We sketch the proof here in order to compare it with the proof of the theorem below, which also uses that method.