ON THE REPRESENTATIONS, $N_3(n^2)$ [1]

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1. **Introduction.** Let the symbol $N_r(n)$ denote the number of representations of the positive integer n in the form $n=x_1^2+x_2^2+\cdots+x_r^2$, where $x_1, x_2, \cdots, x_r$ are positive or negative integers or zero. We will agree to count the two representations $n=x_1^2+x_2^2+\cdots+x_r^2$, $n=y_1^2+y_2^2+\cdots+y_r^2$, as distinct unless simultaneously $x_\nu=y_\nu$, $\nu=1, 2, \cdots, r$. Notice that in a given representation the signs of the roots, as well as their arrangement, are relevant. A zero square, however, is supposed to have only one root.

In a letter written in 1884 to Ch. Hermite, T. J. Stieltjes[2] proved by means of elliptic functions that if $n=p^k$, $p\equiv 1 \pmod 8$, p prime, then $N_3(n^2)=6p^k$. Later in 1907, A. Hurwitz[3] stated without proof that if

$$(1)\qquad n = 2^k m = 2^k PQ, \qquad P = \prod_{\nu=1}^{r} p_\nu^{a_\nu}, \qquad Q = \prod_{\nu=1}^{s} q_\nu^{b_\nu},$$

where each p_ν is a prime $\equiv 1 \pmod 4$, and each q_ν is a prime $\equiv 3 \pmod 4$, then

$$(2)\qquad N_3(n^2) = 6P\prod_{\nu=1}^{s}\left[q_\nu^{b_\nu} + 2\,\frac{q_\nu^{b_\nu}-1}{q_\nu-1}\right].$$

This result is also implicitly contained in Stieltjes' letter mentioned above.

In 1940, G. Pall[4] showed that (2) could be derived arithmetically by an application of certain divisibility properties of the Lipschitz integral quaternions. It is the purpose of this paper to give a simple arithmetical proof of (2) by a method which has been evolved from the study of a paper by Hurwitz[5] in which he derived the analogous formula for $N_5(n^2)$.[6]

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[1] This is the first part of a paper presented to the Society April 6, 1940, under the title *On the number of representations of the square of an integer as the sum of an odd number of squares.*

[2] T. J. Stieltjes, "Lettre 45," *Correspondence d'Hermite et de Stieltjes*, vol. 1, Paris, 1905, pp. 89–94.

[3] A. Hurwitz, *Mathematische Werke*, vol. 2, Basel, 1933, p. 751.

[4] G. Pall, Transactions of this Society, vol. 47 (1940), pp. 487–500. See also G. Pall, Journal of the London Mathematical Society, vol. 5 (1930), pp. 102–105. In this paper Pall gives analytical proofs of the formula for $N_r(cn^2)$, $r=3, 5, 7, 11$, c an integer.

[5] A. Hurwitz, Comptes Rendus de l'Académie des Sciences, Paris, vol. 98 (1884), pp. 504–507; *Mathematische Werke*, vol. 2, pp. 5–7. Notice that Hurwitz makes use of certain results announced by Liouville and some formulas of Stieltjes.

[6] The author wishes to acknowledge the assistance rendered him by Professor J. V. Uspensky.