THE DOUBLE COSETS OF A FINITE GROUP[1]

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1. **Introduction.** It is the purpose of this paper to study some of the properties of the double cosets of a finite group and to prove two main theorems which generalize the results of two previous papers by the author,[2] giving some relations between the double cosets and the irreducible components of the permutation group generated by a given subgroup. We let H be an arbitrary but fixed subgroup of order h of a finite group G of order g, $g=nh$, and we let G_H be the permutation group of degree n induced by right multiplication of the cosets HS_i, $i=1, 2, \cdots, n$, by elements of G. When written as a group of permutation matrices and completely reduced, the group G_H will have r' distinct irreducible components Γ_i of degree n_i and multiplicity μ_i^H, and we may write

$$G_H = \sum_{i=1}^{r'} \mu_i^H \Gamma_i. \tag{1.1}$$

Multiplication of a right coset HG_k on the left by a single element of G does not in general produce a right coset, but if each coset HG_k is multiplied on the left by all the elements of a right coset HS_t and the products are added, a transformation is obtained which carries each of the n right cosets HG_k into a collection of right cosets $\sum_{\theta=1}^{h} HS_tH_\theta G_k$ in which, as we shall see in §4, each of the k_t cosets occurs $d_t=h/k_t$ times. Its matrix d_tV_t is permutable with each of the matrices of G_H. Certain cosets, which we shall call associated cosets, are permuted among themselves when multiplied on the right by elements of H. Each of these produces the same matrix V_t. The totality K_t of elements belonging to a complete set of k_t associated cosets, each counted once, will be called a *double coset*, whereas the term *weighted double coset* will refer to the complex of h^2 elements HS_tH in which each element of the double coset K_t occurs $d_t=n/k_t$ times. The integer d_t will be called the density. The number of distinct double cosets K_t will be denoted by r, and the elements S_t, $t=1, 2, \cdots, r$, one from each, will be said to *generate* the double

[1] Presented to the Society, September 12, 1940.

[2] J. S. Frame, *The degrees of the irreducible components of simply transitive permutation groups*, Duke Mathematical Journal, vol. 3 (1937), pp. 8–17.

J. S. Frame, *On the decomposition of transitive permutation groups generated by the symmetric group*, Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 132–139.