A SEQUENCE OF LIMIT TESTS FOR THE CONVERGENCE OF SERIES[1]

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In this paper, we shall develop a sequence of limit tests for the convergence and divergence of infinite series of positive terms which is similar in form to the De Morgan and Bertrand sequence but involves the ratio of two successive values of the test ratio rather than the test ratio itself. The proof will be based on the following integral test by R. W. Brink:[2]

"THEOREM VI. *Given the sequence* $\{u_n\}$. *Let* $r_n = u_{n+1}/u_n$ *and* $R_n = r_{n+1}/r_n = u_{n+2}u_n/u_{n+1}^2$. *If* $\lim_{n=\infty} r_n = 1$, *and if* $R(x)$ *is a function such that* $R(n) = R_n$, *and such that* $R(x) \geqq R(x')$ *when* $x' > x$, *a necessary and sufficient condition for the convergence of the series* $\sum_{n=0}^{\infty} u_n$ *is the convergence of the integral*

$$\int_0^{\infty} \exp\left\{-\int_0^x \int_x^{\infty} \log R(x)\, dx dx\right\} dx."$$

Since a finite number of terms does not affect convergence or divergence, the conditions of Theorem VI need hold only for n greater than some fixed number ν, in which case zero is to be replaced by ν as a lower limit of integration.

The foregoing theorem admits a generalization similar to that given by C. T. Rajagopal[3] in the case of another theorem of Brink's.[4] However, Brink's Theorem VI is sufficient for the purposes of the present paper.

LEMMA. *Let* $\{u_n\}$ *and* $\{u_n'\}$ *be sequences of positive terms with ratios* $r_n = u_{n+1}/u_n$, $R_n = r_{n+1}/r_n$, $r_n' = u_{n+1}'/u_n'$, *and* $R_n' = r_{n+1}'/r_n'$, *such that* $\lim_{n\to\infty} r_n = \lim_{n\to\infty} r_n' = 1$.

1. *If the series* $\sum_{n=\nu}^{\infty} u_n'$ *converges and if* $R_n \geqq R_n'$ *for all values of* $n \geqq \nu$, *then the series* $\sum_{n=\nu}^{\infty} u_n$ *converges.*

2. *If the series* $\sum_{n=\nu}^{\infty} u_n'$ *diverges and if* $R_n \leqq R_n'$ *for all values of* $n \geqq \nu$, *then the series* $\sum_{n=\nu}^{\infty} u_n$ *diverges.*

[1] Presented to the Society, April 27, 1940, under the title *A sequence of tests for the convergence and divergence of infinite series.*

[2] R. W. Brink, *A new sequence of integral tests for the convergence and divergence of infinite series*, Annals of Mathematics, vol. 21 (1919), pp. 39–60.

[3] C. T. Rajagopal, *On an integral test of R. W. Brink for the convergence of series*, this Bulletin, vol. 43 (1937), pp. 405–412.

[4] R. W. Brink, *A new integral test for the convergence and divergence of infinite series*, Transactions of this Society, vol. 19 (1918), pp. 186–204.