DIFFERENTIAL AND INTEGRAL INVARIANTS OF PLANE CURVES AND HORN ANGLES[1]

J. M. FELD

1. Introduction. A horn angle has been defined by E. Kasner as a figure formed by an ordered pair of oriented analytic arcs which pass through a common point in a common direction and which have contact of order $n \geqq 1$ at the common point. The order of contact is said to be the order of the horn angle. Thus, in a horn angle of the first order the arcs forming its sides have, at the vertex of the angle, the same inclination (direction) but different curvatures. Horn angles of the first order have a unique absolute conformal invariant, first found by Kasner.[2] Two relative conformal invariants of horn angles of the first order were given later in a joint paper by Kasner and Comenetz.[3] Kasner has also determined conformal invariants of the fifth order for horn angles of second order contact.[4]

Kasner and Comenetz obtained the above-mentioned invariants by the use of power series and by repeated differentiation of the Cauchy-Riemann equations. It is shown in §2 of this paper that the relative and absolute conformal invariants of horn angles of the first order can be obtained directly from two standard formulas; one of these expresses d^2y/dx^2 in terms of u and v where $x=x(u)$ and $y=y(v)$; the other expresses the Schwarzian derivative

$$\{y, x\} = \frac{d^3y}{dx^3}\Big/\frac{dy}{dx} - \frac{3}{2}\left(\frac{d^2y}{dx^2}\Big/\frac{dy}{dx}\right)^2$$

in terms of u and v under the same conditions. As a consequence, the invariants are presented in a form which permits of their being interpreted as invariants, not only under the conformal group, but also under a group of multiply differentiable transformations of the type: $x=x(u)$, $y=y(v)$, where x is a function of u alone, and y is a function of v alone.[5]

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[1] Presented to the Society, February 25, 1939.

[2] E. Kasner, *Conformal geometry*, Proceedings of the Fifth International Congress of Mathematicians, Cambridge, 1912, vol. 2, p. 81.

[3] E. Kasner and G. Comenetz, *Conformal geometry of horn angles*, National Academy of Sciences, vol. 22 (1936), p. 303. See also Comenetz, *Kasner's invariant and trihornometry*, American Mathematical Monthly, vol. 45 (1938), p. 28.

[4] Kasner, Transactions of this Society, vol. 44 (1938), p. 25.

[5] Transformations of this type occur, for example, in the papers: Kasner, Transactions of this Society, vol. 16 (1915), pp. 333–349; Cayley, *Collected Works*, vol. 11, p. 148.