REFLEXIVE BANACH SPACES NOT ISOMORPHIC TO UNIFORMLY CONVEX SPACES[1]

MAHLON M. DAY[2]

Clarkson[3] introduced the notion of uniform convexity of a Banach space: B is uniformly convex if for each ϵ with $0<\epsilon\leqq 2$ there is a $\delta(\epsilon)>0$ such that whenever $\|b\|=\|b'\|=1$ and $\|b-b'\|\geqq\epsilon$, then $\|b+b'\|\leqq 2(1-\delta(\epsilon))$. Milman[4] and Pettis[5] have demonstrated that any uniformly convex space is reflexive;[6] that is, that for each $\beta\in B^{**}$ there is a $b\in B$ with $\beta(f)=f(b)$ for every $f\in B^*$. The same result clearly holds if B is not uniformly convex but can be given a new norm defining the same topology under which the space is uniformly convex. It has been conjectured that every reflexive space can be given such a topologically equivalent uniformly convex norm; that is, that, in Banach's terminology,[7] every reflexive space is isomorphic to a uniformly convex space. We shall show by a large class of examples that this is not the case; in fact the following result holds:

THEOREM 1. *There exist Banach spaces which are separable, reflexive, and strictly convex,*[8] *but are not isomorphic to any uniformly convex space.*

We shall start with a class of Banach spaces and pick out a simple example having all but the strict convexity property; with this as a sample of what can happen we easily find a large number of spaces satisfying all the conditions of the theorem. As an application of our results we show that certain ergodic theorems of Alaoglu and Birkhoff[9] can be extended to some non-uniformly convex spaces.

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[1] Presented to the Society, November 23, 1940, under the title *Some more uniformly convex spaces.*

[2] Corinna Borden Keen Research Fellow of Brown University.

[3] J. A. Clarkson, *Uniformly convex spaces*, Transactions of this Society, vol. 40 (1936), pp. 396–414.

[4] D. Milman, *On some criteria for the regularity of spaces of the type* (B), Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS, new series, vol. 20 (1938), pp. 243–246.

[5] B. J. Pettis, *A proof that every uniformly convex space is reflexive*, Duke Mathematical Journal, vol. 5 (1939), pp. 249–253.

[6] That is, "regular" in the terminology of Hahn. We use B^* to denote the conjugate space of the Banach space B; $B^{**}=(B^*)^*$.

[7] S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, chap. 11.

[8] B is strictly convex if the set of points with $\|b\|=1$ contains no line segments; that is, if $\|b\|=\|b'\|=1$ and $b''=tb+(1-t)b'$ with $0<t<1$, then $\|b''\|<1$.

[9] L. Alaoglu and G. Birkhoff, *General ergodic theorems*, Annals of Mathematics, (2), vol. 41 (1940), pp. 293–309.