A NOTE ON A THEOREM BY WITT[1]

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1. **Introduction.** Let F denote the free group with n generators and let F^c be the cth member of the lower central series[2] of F. Witt[3] has shown that $Q^c = F^c/F^{c+1}$ is a free abelian group with $\psi_c(n) = (1/c)\sum \mu(c/d)n^d$ generators (the summation is over all divisors d of c and μ is the Möbius μ-function).

The set of kth powers in F generates a normal subgroup H_k. Let $F_k = F/H_k$ and $G_{k,c} = F_k/F_k^{c+1}$. We shall call F_k the *free k-group* and $G_{k,c}$ the *free k-group of class c*. It is a consequence of Witt's result that F_k^c/F_k^{c+1}, the central of $G_{k,c}$, is abelian and has at most $\psi_c(n)$ generators. In this note we show that if p is a prime greater than c, and $q = p^\alpha$, then the central of $G_{q,c}$ is of order q^N where $N = \psi_c(n)$. If the prime divisors of k are all greater than c, an analogous result holds for the central of $G_{k,c}$ as a consequence of Burnside's theorem that a nilpotent group is the direct product of its Sylow subgroups.

Let M_c denote the space of tensors of rank c over the $GF[p]$. A homomorphic mapping of M_c upon the central of $G_{p,c}$ is set up and enables one to apply the theory of decompositions of tensor space under the full linear group mod p, to determine all characteristic subgroups of $G_{p,c}$ which lie in its central. This theory is applied to determine all the characteristic subgroups of $G_{p,c}$ for $c < 5$ and a multiplication table is constructed for $G_{p,3}$.

2. **Commutator calculus.**[4] Let $s_1, s_2, \cdots$ be operators in any group G and set $s_{12} = (s_1, s_2) = s_1^{-1}s_2^{-1}s_1s_2$ and $s_{12\cdots k} = (s_{12\cdots k-1}, s_k)$. $s_{12\cdots k}$ is called a *simple commutator* of *weight k* in the components $s_1, \cdots, s_k$. The group G^k generated by the simple commutators of weight k for all choices of $s_1, \cdots, s_k$ in G is called the kth member of the *lower central series* of G. If $s \in G^k$ but $s \notin G^{k+1}$, then s is said to have *weight k* in G.

For all s_1, s_2, s_3 in G we have

$$(1) \qquad (s_1s_2, s_3) = s_{13}s_{132}s_{23}, \qquad (s_1, s_2s_3) = s_{13}s_{12}s_{123}.$$

Let the weight of s_i be α_i and set $\alpha = \alpha_1 + \cdots + \alpha_k + 1$. The following relations are then true:

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[1] Presented to the Society, April 13, 1940.

[2] For definition see §2 below or [**4**, p. 49].

[3] [**7**, p. 153].

[4] The relations in this section are either taken directly from Hall, Magnus, or Witt or are immediate consequences of their theorems. See [**4**, **6** and **7**].