RECURRENCE OF SYMBOLIC ELEMENTS IN DYNAMICS[1]

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1. **Introduction.** Morse and Hedlund [1] have given a symbolic treatment of modern theoretical dynamics as developed by Birkhoff [2] and others. In the Morse-Hedlund viewpoint the concept of recurrence plays an important role. To establish various theorems on symbolic trajectories Morse and Hedlund introduced "symbolic elements," the analogues of line elements on ordinary trajectories, a symbolic element being the notion of a trajectory T and a particular symbol in T. In the present paper we shall be concerned primarily with the question: *"How are recurrence of a trajectory T and elements based on T related?"*

2. **Definitions.** For terms defined elsewhere and used here the reader is referred to other papers [**1**, **3**]. Let rays R_i $(i=1, 2, 3, 4)$ be given by

$$\alpha_{i1}\ \alpha_{i2}\ \alpha_{i3} \cdots .$$

The distance E_1E_2 between the elements $E_1=(R_1, R_2)$ and $E_2=(R_3, R_4)$ is defined to be $1/n$ where n is such that $a_{1j}=a_{3j}$, $a_{2j}=a_{4j}$ for each value of j in the range $1, 2, \cdots, n$, while

$$(a_{1,n+1},\ a_{2,n+1}) \neq (a_{3,n+1},\ a_{4,n+1}).$$

The element E_1 is the ray

$$A_1A_2A_3 \cdots , \tag{1}$$

where A_j denotes the pair of symbols (a_{1j}, a_{2j}). In what follows the term "recurrence of E_1" designates the recurrence of the ray (1). We recall that a ray (1) is *recurrent* if for each n there is an m such that each n-block in (1) is contained in each m-block of (1). For each n the value $R(n)$ of the *recurrency* function of (1) is the smallest m with the property just mentioned.

3. **Recurrence.** It is evident that the recurrence of a single element based on a trajectory T does not imply the recurrence of T.

THEOREM 1. *If each element based on a trajectory T is recurrent, T is recurrent.*

The recurrence of each ray based on T obviously implies the recur-

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