NOTE ON AUTOPOLAR CURVES[1]

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1. **Introduction.** The aim of this paper is to study those curves which are autopolar with respect to the parabola $2\eta=\xi^2$. The method, which is believed to be new, is to consider these curves as special solutions of those differential equations which are invariant under the dual substitutions for the above conic of reference.[2] It will be obvious that this method may be readily modified for the study of curves which are autopolar with respect to any conic. The parabola $2\eta=\xi^2$ has been chosen for the sake of the simplicity of the substitutions.

2. **Dual substitutions for the conic of reference.** In the ordinary differential equation,

$$f(x, y, y', y'', \cdots, y^{(n)}) = 0, \tag{1}$$

let us make the well known dual substitutions[3]

$$\begin{aligned} x = P = Y', \qquad y = Y'X - Y, \qquad p = y' = X,\\ y'' = 1/Y'', \ y''' = - Y'''/Y''^3, \cdots \end{aligned} \tag{2}$$

for which the conic of reference is the parabola $2\eta=\xi^2$. We obtain a new differential equation,

$$f(Y', Y'X - Y, X, 1/Y'', \cdots) = 0, \tag{3}$$

whose solution is, let us say,

$$\phi(X, Y, c_1, c_2, \cdots, c_n) = 0. \tag{4}$$

If we eliminate X, Y from equation (4) and the following two equations,[4]

$$x\frac{\partial\phi}{\partial Y} + \frac{\partial\phi}{\partial X} = 0, \qquad -y\frac{\partial\phi}{\partial Y} = Y\frac{\partial\phi}{\partial Y} + X\frac{\partial\phi}{\partial X}, \tag{5}$$

we shall have the solution of the original differential equation (1), which we shall denote by

$$F(x, y, c_1, c_2, \cdots, c_n) = 0. \tag{6}$$

3. **Geometrical interpretation.** Let C be any curve of the family

[1] Presented to the Society, October 28, 1939.

[2] A. R. Forsyth, *A Treatise on Differential Equations*, 3d edition, 1903, pp. 45–47.

[3] Forsyth, op. cit.

[4] Forsyth, loc. cit.