DIVISORS OF ZERO IN MATRIC RINGS

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1. **Introduction.** An element a of a ring S is a *divisor of zero* in S if there exists a nonzero element x of S such that $ax=0$, or a nonzero element y of S such that $ya=0$. The purpose of the present note is to obtain several theorems about divisors of zero in matric rings which, although quite elementary in character, have apparently not been previously noted. Throughout, unless otherwise stated, R will be used to denote an arbitrary commutative ring with unit element. Let R_n denote the ring of all matrices of order n with elements in R, and $R[\lambda]$ the ring of polynomials in the indeterminate λ with coefficients in R. If A is an element of R_n, we shall denote by $f(\lambda)=|\lambda-A|$ the *characteristic polynomial* of A, and thus $f(\lambda)$ is an element of $R[\lambda]$, with leading coefficient 1. The ideal $\mathfrak{m}$ of all elements $g(\lambda)$ such that $g(A)=0$ is the *minimum ideal* of A. If the minors of $\lambda-A$ of order $n-1$ are denoted by $h_{ij}(\lambda)$ $(i, j=1, 2, \cdots, n)$, it has been shown in a previous paper[1] that $g(\lambda)\equiv 0$ $(\mathfrak{m})$, if and only if

$$g(\lambda)h_{ij}(\lambda) \equiv 0 \quad (f(\lambda)), \qquad i, j = 1, 2, \cdots, n.$$

If $R[A]$ denotes the subring of R_n generated by A together with the unit element of R_n, which we identify with the unit element of R, then the elements of $R[A]$ are the polynomials in A with coefficients in R. It is quite easy to show that A is a divisor of zero in R_n if and only if $|A|$ is a divisor of zero in R. But we shall show, in §2, that A is actually a divisor of zero in $R[A]$ if it is a divisor of zero in R_n—a fact which is almost trivial if R is a field. This theorem is used in the following section in which we define, by means of the Sylvester determinant, the *resultant* $\mathcal{R}(f, g)$ of two elements $f(\lambda)$ and $g(\lambda)$ of $R[\lambda]$ and show, following Frobenius, that if $f(\lambda)$ has leading coefficient unity,

$$\mathcal{R}(f, g) = |\, g(A)\,|,$$

where A is *any* matrix having $f(\lambda)$ as characteristic polynomial. It then follows readily that an element $g(\lambda)$ of $R[\lambda]$ is *prime to*[2] $\mathfrak{m}$ if and

[1] Neal H. McCoy, *Concerning matrices with elements in a commutative ring*, this Bulletin, vol. 45 (1939), pp. 280–284. Hereafter, this paper will be referred to as M.

[2] For definitions of this term, see W. Krull, *Idealtheorie in Ringen ohne Endlichkeitsbedingung*, Mathematische Annalen, vol. 101 (1929), pp. 729–744. This will be referred to later as K. A definition will also be found in §3 of the present paper.