NOTE ON A CERTAIN TYPE OF DIOPHANTINE SYSTEM[1]

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1. **The quadratic case.** Several writers[2] have considered the problem of making a^2x^2+dx, b^2x^2+ex, $\cdots$ simultaneously squares, a, d, b, e, $\cdots$ being constants, without discussing the conditions under which an integer solution exists. The fact that the coefficients of x^2 are squares is of no consequence. It is shown in §2 how necessary and sufficient conditions for the existence of integer solutions in much more general problems (§2, (9), (13)) can be determined and how, when these conditions are satisfied, the solutions may be found. The details are given first for three very special cases, (1), (5), (6) below.

If all letters denote integers, and a, b, c, d are constant, we seek necessary and sufficient conditions that the system

$$ax^2 + bx = y^2, \qquad cx^2 + dx = z^2 \tag{1}$$

shall have a solution x, y, z. We shall assume that $abcd \neq 0$, as the excluded possibilities require only slight modifications, all of which are included in the general method.

The required conditions are that b, d be simultaneously representable in two forms of degree seven. Precisely, (1) has a solution in integers x, y, z if and only if integers f, g, h, k, m, n exist such that

$$b = hf^2(m^2 - ag^2k^2), \qquad d = hg^2(n^2 - cf^2k^2). \tag{2}$$

Provided such f, g, $\cdots$ exist, the complete solution of (1) is

$$x = hf^2g^2k^2, \qquad y = hf^2gkm, \qquad z = hfg^2kn,$$

where f, g, h, k, m, n run through all solutions of (2).

To prove this, we rewrite (1) as

$$x(ax + b) = y^2, \qquad x(cx + d) = z^2,$$

which is of the form

$$xu = y^2, \qquad xv = z^2, \tag{3}$$

with

$$u \equiv ax + b, \qquad v \equiv cx + d.$$

The complete integer solutions of the respective equations in (3),

[1] Presented to the Society, April 6, 1940.

[2] Summary of results to 1920 in L. E. Dickson, *History of the Theory of Numbers*, vol. 2, 1920, chap. 18.