ON REGULAR FAMILIES OF CURVES[1]

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A family F of non-intersecting curves filling a metric space is called *regular* if, in a neighborhood of any point p, it is homeomorphic with a family of straight lines. We have given in another paper[2] a necessary and sufficient condition, which we shall call (A′) (to be described below), that a family F be regular. We shall prove in this note that the following condition is sufficient:

(A) Given any point p, and a direction on the curve through p, there is an arc pq in this direction with the following property. For every $\epsilon>0$ there is a $\delta>0$ such that for any p', with $\rho(p', p)<\delta$, there is an arc $p'q'$ of $C(p')$ such that

$$(1) \qquad p'q' \subset V_\epsilon(pq), \qquad q' \subset V_\epsilon(q).$$

The condition (A′) is the same, except that after (1), we add:

(2) If r' and s' are on $p'q'$ and $\rho(r', s')<\delta$, then $\delta(r's')<\epsilon$.

From the present theorem it is clear that the families of curves recently defined by Niemitzki[3] are regular.

To prove the theorem, suppose (A) holds, but (A′) does not. Then the following is true:

(B) There is a point p, and a direction of the curve $C(p)$, such that for any arc pq on $C(p)$ in this direction, there is an $\epsilon>0$, such that for any $\delta>0$, there is a point p', with $\rho(p', p)<\delta$, such that for any q' on $C(p')$,

(3) either $p'q' \not\subset V_\epsilon(pq)$, or $q' \not\subset V_\epsilon(q)$,

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[1] Presented to the Society, April 27, 1940.

[2] Annals of Mathematics, (2), vol. 34 (1933), pp. 244–270. We refer to this paper as RF. By RF, Theorem 7A, F is regular as there defined. The converse is proved as follows. By Theorem 17A, there is a cross-section S through p. In a neighborhood of p, the curves are orientable (this is easily seen, for instance, with the help of Theorem 9B). Choose an open subset S' of S, and let U be all points $q'=g'(q, \alpha)$, q in S', $|\alpha|<\epsilon$ (see RF, §15); U is a neighborhood of p, expressed as the product of S' and the open line segment $-\epsilon<\alpha<\epsilon$.

By a curve, we shall mean here the topological image of an open line segment or of a circle. We shall use $\rho(p, q)$ for distance, $\delta(A)$ for the diameter of the set A, and $V_\epsilon(A)$ for the set of all points p, $\rho(p, A)<\epsilon$. Let $C(p)$ mean the curve of F through p.

[3] V. Niemytzki, Recueil Mathématique de Moscou, vol. 6 (48) (1939), pp. 283–292. We mention two further papers in the subject: H. Whitney, Duke Mathematical Journal, vol. 4 (1938), pp. 222–226, showing that if the curves fill a region in 3-space, a cross-section may be chosen so as to be a 2-cell; W. Kaplan, Duke Mathematical Journal, vol. 7 (1940), pp. 154–185, studying families filling the plane.