

PROOF OF A THEOREM OF HALL

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In the Journal of the London Mathematical Society for July, 1937, Mr. Philip Hall gave a proof of the theorem, "If a group G of order g has a subgroup of order m for every divisor m of g such that $(m, g/m) = 1$, then G is a soluble group." The proof is a very simple one in contrast to the rather difficult proof of the converse theorem which Hall had published in the same journal for April, 1927. It seems worthwhile to give a simpler proof of this converse.

THEOREM. *If g is the order of a soluble group G and m is a divisor of g such that m and g/m are relatively prime, then G has a subgroup of order m and furthermore all the subgroups of order m in G are conjugate under G .*

PROOF. Since the theorem is true by default for prime power groups, let us suppose that it is true for all soluble groups of orders less than g and use the method of complete induction. Then the theorem is true for an invariant subgroup G' of prime index r in G .

If r divides g/m , then G' has a subgroup of order m . Since G' is invariant and $(m, r) = 1$, every element of order dividing m in G must be in G' . Hence all the subgroups of order m in G must be in G' where by hypothesis they form a complete set of conjugates under G' and consequently a complete set of conjugates under G .

If r divides m , then G' has a complete set of conjugates of order m/r . Since G' is invariant, the subgroups of order m/r in G' are a complete set of conjugates under G . Let M' be one of these subgroups of order m/r in G' .

If M' is the only subgroup of order m/r in G' , then, it must be invariant in G . Then the quotient group is of order rg/m and since r is a Sylow divisor of the order of this quotient group, it has a subgroup of order r . Then G has a subgroup of order $r \cdot m/r$ or m . On the other hand if M' is not the only subgroup of order m/r in G' , let it be one of k subgroups of order m/r in G' . Then k divides the order g/r of G' and the normalizer of M' in G is of order g/k divisible by m . Since this normalizer is a soluble group of order g/k less than g , it has a subgroup of order m .

There remains only to show, for the case r divides m , that all the subgroups of order m in G are conjugate under G . Let M be a subgroup of order m . If there is no other subgroup of order m in G , then the theorem is true by default. However, if M_1 is another subgroup