NOTE ON NORMALITY IN QUASI-GROUPS

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An interesting problem in the theory of quasi-groups is to determine how strong an associative law must be assumed in order to obtain a theory of normal subquasi-groups similar to that of ordinary groups. The properties of normal subgroups which it is desirable to retain in the non-associative case are, first, that they form a Dedekind structure, and second, that each one gives rise to a quotient group homomorphic to the whole group. In this note we shall take the latter property as the definition of normality and show that the former follows from it under very general conditions.

We shall understand by a quasi-group G a system of elements satisfying the following two postulates:

I. Product axiom. *Any ordered pair of elements a, b of G has a unique product ab which also belongs to G.*

II. Quotient axiom. *For any two elements a and b of G there exist unique elements x and y of G such that*

$$ax = b, \qquad ya = b.$$

We shall make one further assumption, namely that there exist left coset expansions with respect to any subquasi-group H of G. In other words, if H is any subquasi-group, the cosets aH and bH are either identical or have no elements in common. This assumption has been shown by Hausmann and Ore [1] to be equivalent to the following:

III. Weak associative law. *If a and b are arbitrary elements of G and c_0 a fixed element, let d_0 be determined so that $(ab)c_0 = ad_0$. Then for any c*

$$(ab)c = ad,$$

where d belongs to the quasi-group $\{c_0, d_0, c\}$ generated by c_0, d_0 and c.

Since no confusion can arise, we shall for convenience use the term subgroup for subquasi-group and quotient group for quotient quasi-group, without implying thereby that the systems in question are associative.

It should be noted that a theory of normal subgroups has been given by Hausmann and Ore [1] based on the definition of a normal subgroup as a subgroup H such that $aH = Ha$ for all a in G. This