SOME NEW PROPERTIES OF TRANSFINITE ORDINALS[1]

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1. **Introduction.** Sierpiński[2] has stated the following theorem on the Cantor normal form:

Every ordinal $A > 0$ can be represented in the form

$$A = \sum_{i=0}^{n-1} \omega^{\alpha_i} a_i,$$

where

$$\omega > n, a_0, a_1, a_2, \cdots, a_{n-1} > 0,$$
$$A \geqq \alpha_0 > \alpha_1 > \alpha_2 > \cdots > \alpha_{n-1} \geqq 0.$$

Henceforth in this paper, when an ordinal is written in summation form, it will be assumed that the summation satisfies the above-stated requirements.

Elsewhere Sierpiński[3] has proved:

The set of all "divisors on the left" of an ordinal number is closed.[4]

One of the purposes of this paper is to analyze further, by means of the Cantor normal form, the left and right factors of transfinite ordinals. In addition the Cantor normal form will be used to prove that for ordinals A, B, and C

$$(A + B)C \leqq AC + BC.$$

We can also specify the necessary and sufficient condition that the equality hold. F. Siecza[5] has used the Cantor normal form to discuss

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[2] W. Sierpiński, *Leçons sur les Nombres Transfinis*, Paris, 1928, p. 202; hereafter referred to as Sierpiński (I). The theorem was first stated for ordinals of the first and second ordinal class by G. Cantor, *Beiträge zur Begründung der transfiniten Mengenlehre*, Mathematische Annalen, vol. 49 (1897), p. 237, and for all ordinals by G. Hessenberg, *Grundbegriffe der Mengenlehre*, Abhandlungen der Fries'schen Schule, New Series 1.4, Göttingen, 1906, p. 587.

[3] W. Sierpiński, *A property of ordinal numbers*, Bulletin of the Calcutta Mathematical Society, vol. 20 (1930), pp. 21–22; hereafter referred to as Sierpiński (II).

[4] A set of ordinal numbers is said to be closed if it contains the least upper bound of each subset.

[5] F. Siecza, *Sur l'unicité de la décomposition de nombres ordinaux en facteurs irreductibles*, Fundamenta Mathematicae, vol. 5 (1924), pp. 172–175.