THE GENERALIZATION OF A LEMMA OF M. S. KAKEYA

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We shall prove the following:

LEMMA. *It is always possible to find the unique polynomial*

$$\phi^*(z) = \sum_{k=0}^{2s} \gamma_k^* z^k$$

of degree $2s$ *possessing the following properties:*

I. $$\phi^*(z) = ci^2(z)\tau(z)\tau^*(z), \qquad c = \text{const.},$$

the polynomial $i(z)$ *of degree* $\sigma \leqq s$ *having all roots in the domain* $|z| > 1$:

$$i(z) = \prod_{i=1}^{\sigma} (z - a_i), \qquad |a_i| > 1, \quad i = 1, 2, \cdots, \sigma,$$

and the polynomial $\tau(z)$ *being of degree* $\nu = s - \sigma$:

$$\tau(z) = \prod_{i=1}^{\nu} (z - \alpha_i), \qquad \tau^*(z) = z^\nu \bar{\tau}\left(\frac{1}{z}\right) = \prod_{i=1}^{\nu} (1 - z\bar{\alpha}_i).$$

II. *It is subject to the conditions*

$$\omega_i(\phi^*) = \sum_{k=0}^{2s} \gamma_k^* c_k^{(i)} = d_i, \qquad i = 0, 1, \cdots, s,$$

the given linear functionals ω_i *being such that every polynomial* $\phi(z)$ *of degree* $n \geqq 2s$ *for which*

$$\omega_i(\phi) = \sum_{k=0}^{2s} \gamma_k c_k^{(i)} = 0, \quad (i = 0, 1, \cdots, s), \qquad \phi(z) = \sum_{k=0}^{n} \gamma_k z^k,$$

has $s+1$ *roots at least in the domain* $|z| < 1$.

In the particular case when

$$\omega_i(\phi) = \phi^{(i)}(z_k), \qquad |z_k| < 1,$$

this lemma has been proved by M. S. Kakeya [1];[1] without being aware of his result we have proved this lemma in the case[2]

[1] Numbers in brackets refer to the bibliography at the end.

[2] In [1] and [2] one may find the application of this lemma to some extremal problems.